Statistisk sentralbyrå
Statistics Norway

A novel multivariate composite estimator for the Labour Force Survey

Håvard Hungnes

TALL SOM FORTELLER

DISCUSSION PAPERS

1022

Discussion Papers: comprise research papers intended for international journals or books. A preprint of a Discussion Paper may be longer and more elaborate than a standard journal article, as it may include intermediate calculations and background material etc.

The Discussion Papers series presents results from ongoing research projects and other research and analysis by SSB staff. The views and conclusions in this document are those of the authors.

Published: April 2025

Abstracts with downloadable Discussion Papers
in PDF are available on the Internet:

https://www.ssb.no/discussion-papers
http://ideas.repec.org/s/ssb/dispap.html

ISSN 1892-753X (electronic)

Abstract

This paper introduces a novel multivariate composite estimator for the Labour Force Survey (LFS). Unlike the univariate composite estimators used in some countries, the multivariate estimator takes into account the different probabilities of transitioning between labour market categories, such as employment, unemployment, or non-participation in the labour force. By directly estimating all categories, it avoids residual determination issues, where one category is estimated as the difference between the total population and the sum of others. The multivariate approach improves the accuracy of the population estimates for each category. Additionally, the paper introduces a method to account for time-varying biases associated with how long the respondents have participated in the survey, explicitly incorporating wave-specific effects and their evolution over time.

Keywords: Autocorrelated sampling errors; Labour Force Survey (LFS); Multivariate composite estimator; Population estimation accuracy; Survey wave-specific biases; Transition probabilities in labour market categories.

JEL classification: C13; C83; J21.

Acknowledgements: Thanks to Pål Boug, Jørn Ivar Hamre, and Terje Skjerpen, and participants at the *Econometric Models and Economic Policy Analysis workshop* at BI Norwegian Business School in October 2023 and *The 2024 RCEA International Conference in Economics, Econometrics, and Finance* at Brunel University London in May 2024 for valuable comments.

Address: Håvard Hungnes, Statistics Norway, Research Department.
E-mail: havard.hungnes@ssb.no

Sammendrag

Artikkelen presenterer en ny multivariat sammensatt estimator for Arbeidskraftundersøkelsen (AKU). I motsetning til univariate estimatorer som brukes i enkelte land, tar denne multivariate tilnærmingen hensyn til de ulike overgangssannsynlighetene mellom arbeidsmarkedsstatusene: sysselsatt, arbeidsledig og utenfor arbeidsstyrken. Ved å estimere alle kategorier direkte, unngås problemer knyttet til residualbestemmelse, noe som forbedrer nøyaktigheten av populasjons-estimatene.

En viktig forbedring som foreslås i dokumentet er å ta høyde for potensielt tidsvarierende skjevheter i svarene fra respondenter, spesielt med hensyn til hvordan svarene varierer mellom ulike undersøkelsesbølger ("wave-specific effects"). Tidligere estimatorer har ofte oversett slike skjevheter. Ved å inkludere en metode for å justere for disse skjevhetene, gir den nye estimatoren mer pålitelige arbeidsmarkedstall.

Empiriske resultater fra anvendelse på den norske AKU viser at en innføring av den multivariate estimatoren fører til relativt små endringer i arbeidsledighetstallene sammenlignet med en direkte estimator, men gir betydelige revisjoner av sysselsettingsestimatene. Særlig yngre aldersgrupper viser store avvik, noe som understreker viktigheten av å modellere sammenhenger mellom ulike arbeidsmarkedsstatuser eksplisitt.

Den nye estimatoren bygger videre på tidligere forskning, men introduserer en mer fleksibel tilnærming som tillater tidsvariasjon i undersøkelsesbølgenes skjevheter. Dette gir statistiske myndigheter et verktøy for å forbedre nøyaktigheten i arbeidsmarkedsstatistikken.

1 Introduction

The Labour Force Survey (LFS) provides essential data on the employment status of individuals, including information on employment, unemployment, and non-participation in the labour force. It also provides details into occupations, working hours, and the duration of job search for the unemployed.

Many countries use a rotating panel design for the LFS, where participants are surveyed multiple times and a portion of the sample is replaced in each round. Although this design offers several advantages, it also introduces challenges, such as possible autocorrelation in the survey errors. If not properly taken into account, autocorrelation can lead to misleading trends, as highlighted by Mayer (2018) and Hausman and Watson (1985).

One approach to addressing this issue is to use a composite estimator, which combines a 'direct' estimator with one that accounts for autocorrelation due to the rotating sample. While univariate composite estimators are used for the employment and unemployment figures in the U.S. LFS, they present certain limitations. Specifically, univariate frameworks often result in asymmetrical treatment of a residual category. A multivariate composite estimator resolves this issue.

Another advantage of applying a multivariate composite estimator is that it takes into account different transition probabilities between different categories. For instance, if employment levels are low in a specific sub-group of the sample, a multivariate estimator can distinguish whether this is due to higher unemployment or greater non-participation in the labour market. A univariate estimator will ignore such information as it typically differentiates only between employed and non-employed individuals.

Previous research has shown that different biases can arise depending on how long an individual has been in the survey. For example, first-time participants often report higher unemployment levels. Univariate composite estimators suggested in the literature have typically ignored these biases. This paper introduces a multivariate composite estimator that allows for these biases, referred to as wave-specific effects or wave-specific biases. Moreover, it presents a framework for allowing these wave-specific effects to evolve over time. We derive a simple formula for updating these wave-specific effects, which national statistical offices can easily incorporate into statistical production.

To illustrate the usefulness of the multivariate composite estimator, we apply it to the Norwegian LFS. Our empirical results show that revisions to unemployment estimates are relatively small when switching from the direct estimator to the multivariate composite estimator. However, significant revisions are observed for employment figures, especially for demographic groups with younger individuals, reflecting the benefits of explicitly modelling interdependence between labour market statuses.

The multivariate composite estimator proposed here shares several features with the estimator suggested by Merkouris (2024). While the estimator in Merkouris (2024) accounts for the weighting of the survey respondents, it does not incorporate wave-specific biases. In contrast, the estimator presented here explicitly accounts for the presence of wave-specific biases and allows these effects to evolve over time. This distinction is a key feature of our approach, enabling more accurate modelling of survey data.

The remainder of the paper is structured as follows. Section 2 reviews the related literature. Section 3 describes the rotation patterns used in LFS surveys across various countries. Section 4 revisits the univariate composite estimator, while Section 5 introduces its multivariate counterpart. Section 6 discusses how to estimate the parameters of the multivariate composite estimator. Section 7 addresses the potentially time-varying nature of the wave-specific effects. Section 8 applies the proposed methodology to the Norwegian LFS. Finally, Section 9 concludes the paper.

2 Related literature

There is a long tradition of applying time series models to LFS data in order to improve population estimates. Direct survey estimates can be decomposed into a population estimate and a survey error. The foundation for composite estimators in repeated surveys is laid by Jessen (1942) and Patterson (1950), who utilized generalized least squares methods. The primary objective of the composite estimator is to correct for autocorrelation in survey errors that arise due to the rotating panel design of the survey.

The composite estimator proposed by Hansen et al. (1955) accounts for the autocorrelation in survey errors due to the rotating panel design. This estimator has been used in the U.S. Current Population Survey (CPS), the U.S. variant of the LFS. Using information from previous survey periods, the estimator improves the population estimates for the current period. The enhanced version of this composite estimator, suggested by Gurney and Daly (1965) and evaluated by Breau and Ernst (1983), remains the basis for the published U.S. labour force statistics at the federal level (see U.S. Census Bureau, 2019).

Composite estimators can also be derived in conjunction with the weighting of survey participants. The early contributions of Fuller (1990) and Singh and Merkouris (1995) laid the foundations for such estimators. These methods have been applied in the Canadian LFS, as demonstrated in Singh et al. (1997, 2001), and Fuller and Rao (2001). Recently, Merkouris (2024) proposed a multivariate composite estimator that incorporates individual weighting into the survey design, advancing the methodological frontier.

Time series models have also been used to model population estimates directly, starting with the foundational work by Scott and Smith (1974) and Scott et al. (1977). Pfeffermann (1991) and Pfeffermann et al. (1998) further advanced this approach by introducing a structural model that decomposes population estimates into trend, seasonal, and irregular components, while also accounting for autocorrelation in survey errors across waves caused by rotating panel designs. These time series models can be improved through the integration of auxiliary information, as demonstrated in studies by Harvey and Chung (2000), van den Brakel and Krieg (2016), and Schiavoni et al. (2021, 2024).

The survey sample can be divided into different waves based on the length of time that interviewees have participated in the survey. Studies by Stephan et al. (1954), Hansen et al. (1955), and Bailar (1975) identified clear evidence of wave-specific biases in the U.S. CPS. Specifically, they found that first-time participants (i.e., the first wave) tend to report higher unemployment rates than those who participated in subsequent waves. To account for this, Pfeffermann (1991) incorporated time-invariant wave-specific effects into the structural model. However, Krueger et al. (2017) showed that these biases can evolve over time, suggesting that structural models should allow for time-varying wave-specific effects. This approach is used in van den Brakel and Krieg (2009) and further developed in Hungnes et al. (2024).

Silva and Smith (2001) proposed a time series model for labour market categories (e.g., employed, unemployed, and non-participation) that allows for simultaneous modelling of mutually exclusive categories. Their formulation ensures that it does not matter which labour market category is determined residually.[1]

In situations with large fluctuations in the labour market, such as during the COVID-19 pandemic, it can be problematic to assume that the population variables follow a time-invariant process. The sharp increase in unemployment during the beginning of the pandemic, for example, underscored the need for flexible models. In response, van den Brakel et al. (2022) proposed a flexible trend model to account for rapid population changes, applying it to the Dutch LFS. Similarly, Gonçalves et al. (2022) employed flexible trend models to the Brazilian LFS, and Hungnes et al. (2024) utilized such modells to estimate redesign breaks in the Norwegian LFS during the pandemic.

Given that the process of population estimates can change over time — particularly during periods of economic shocks such as the recent COVID-19 pandemic — using a composite estimator can be advantageous. The composite estimator can be further refined by incorporating wave-specific effects and by simultaneously modelling different labour market categories. This paper proposes a multivariate composite estimator that addresses these ideas.

3 LFS around the world

Many countries employ a rotating panel design in their LFS. In the U.S. LFS, known as the Current Population Survey (CPS), a 4-8-4 rotating panel system is used. Under this system, participants are interviewed for four consecutive months, followed by an eight-month gap, after which they are

[1] A similar joint model is discussed in van den Brakel and Roels (2010).

Table 1: The rotation pattern in the U.S. LFS (Current Population Survey, CPS)

first period	$t =$																							
in sample	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$t = -14$	w4																							
$t = -13$	w3	w4																						
$t = -12$	w2	w3	w4																					
$t = -11$	w1	w2	w3	w4																				
$t = -10$		w1	w2	w3	w4																			
$t = -9$			w1	w2	w3	w4																		
$t = -8$				w1	w2	w3	w4																	
$t = -7$					w1	w2	w3	w4																
$t = -6$						w1	w2	w3	w4															
$t = -5$							w1	w2	w3	w4														
$t = -4$								w1	w2	w3	w4													
$t = -3$									w1	w2	w3	w4												
$t = -2$	w4									w1	w2	w3	w4											
$t = -1$	w3	w4									w1	w2	w3	w4										
$t = 0$	w2	w3	w4									w1	w2	w3	w4									
$t = 1$	w1	w2	w3	w4									w1	w2	w3	w4								
$t = 2$		w1	w2	w3	w4									w1	w2	w3	w4							
$t = 3$			w1	w2	w3	w4									w1	w2	w3	w4						
$t = 4$				w1	w2	w3	w4									w1	w2	w3	w4					
$t = 5$					w1	w2	w3	w4									w1	w2	w3	w4				
$t = 6$						w1	w2	w3	w4									w1	w2	w3	w4			
$t = 7$							w1	w2	w3	w4									w1	w2	w3	w4		
$t = 8$								w1	w2	w3	w4									w1	w2	w3	w4	
$t = 9$									w1	w2	w3	w4									w1	w2	w3	w4

The first column indicates when the sample is included in the survey for the first time. The row shows which wave and block the participants belong to in different survey periods. Yellow colour indicates that the participants are in block 1, while green colour indicates block 2. 'w1', 'w2', 'w3', and 'w4' indicate that the participants belong to waves 1, 2, 3, or 4 of this block in the relevant survey period.

re-interviewed for another four months. This system is displayed in Table 1, where the first set of interviews is referred to as 'Block 1' (the initial four months of participation) and marked in yellow, and the second set as 'Block 2' (the last four months) and marked in green. Within each block, each interview period is referred to as a 'wave', labeled as 'w1', 'w2', 'w3', and 'w4', corresponding to each of the four months in the block.

To illustrate this further, consider the row for $t = 1$ in Table 1, which shows the interview schedule for participants entering the survey at period 1. Initially, these participants are classified as wave 1 in block 1. Over the next three periods (2, 3, and 4), they progress through waves 2, 3, and 4 in block 1, respectively. After an eight-month absence from the survey, they rejoin in period 13, now as wave 1 in block 2, with subsequent waves 2, 3, and 4 in periods 14, 15, and 16, respectively.

Column 1 in Table 1 outlines the participants interviewed in each period. Those participating for the first time are wave 1 in block 1. Those who participated in the survey for the first time in period 0 constitute wave 2 in block 1 in period 1. In period 1, those who joined the survey for the first time in period -1 and -2 now constitute wave 3 and wave 4, respectively, of block 1. The breakdown for block 2 follows a similar pattern, with participants from periods -11, -12, -13, and -14 forming waves 1-4 of block 2 in period 1.

This structure of the U.S. LFS, where survey participants are interviewed across two blocks, highlights the necessity of a composite survey estimator that accounts for autocorrelation of survey errors within each block. However, this study assumes that autocorrelation diminishes significantly between blocks due to the time gap, and thus does not consider correlations across blocks.

The key information needed from the LFS designs of various countries is the number of blocks, the number of waves per block, and the frequency of interviews within each block. For instance, in the U.S., interviews occur monthly within each block, while in Eurostat countries, they take place

Table 2: LFS in some non-Eurostat countries

Country	$\sum_{i=1}^{B} J^i$	l	source
Australia	8	1	Zhang et al. (2019)
Brazil	5	3	Gonçalves et al. (2022)
Canada	8	1	Statistics Canada (2017)
Japan	4+4	1	Statistics Bureau of Japan (2023)
New Zealand	8	3	Stats NZ Tatauranga Aotearoa (2023)
United Kingdom	5	3	Eurostat (2014)
United States	4+4	1	U.S. Census Bureau (2019)

J^i is the number of waves in each block; B is the number of blocks, so $J = \sum_{i=1}^{B} J^i$ is the overall number of waves over all blocks; l is the time between re-interviewing (measured in months).

Table 3: LFS in the Eurostat countries

Country	$\sum_{i=1}^{B} J^i$	l	Country cont.	$\sum_{i=1}^{B} J^i$	l
Austria	5	3	Lithuania	2+2	3
Belgium	2+2	3	Luxembourg	5	3
Bulgaria	2+2	3	Malta	2+2	3
Croatia	2+2	3	Netherlands	5	3
Cyprus	6	3	Montenegro	2+2	3
Czechia	5	3	North Macedonia	2+2	3
Denmark	2+2	3	Norway	8	3
Estonia	2+2	3	Poland	2+2	3
Finland	3+2	3	Portugal	6	3
France	6	3	Romania	2+2	3
Germany	2+2	3	Serbia	2+2	3
Greece	6	3	Slovakia	5	3
Hungary	6	3	Slovenia	3+2	3
Iceland	3+2	3	Spain	6	3
Ireland	5	3	Sweden	8	3
Italy	2+2	3	Switzerland	2+2	3
Latvia	2+2	3	Türkiye	2+2	3

J^i is the number of waves in each block; B is the number of blocks, so $J = \sum_{i=1}^{B} J^i$ is the overall number of waves over all blocks; l is the time between re-interviewing (measured in months). Source: Eurostat (2022).

every three months. These differences are crucial for constructing a composite estimator.

Table 2 summarizes the LFS design features of selected non-Eurostat countries, while Table 3 provides an overview of designs used in Eurostat countries.

In Eurostat countries, the LFS is covering all weeks within a quarter (see Eurostat, 2022). Participants are surveyed every 13 weeks, as noted by Elliott and Zong (2019). Both quarterly and monthly LFS statistics are produced. For monthly statistics, each month consists of four or five survey weeks, and participants are generally interviewed once every three months. For further discussion on the advantages and disadvantages of different rotation patterns, see Steel and McLaren (2009).

4 The univariate composite estimator revisited

In this section, we will consider the univariate composite estimator. In Section 4.1, we assume that the LFS survey consists of only one block. The case where the LFS survey consists of multiple blocks is considered in Section 4.2.

In the LFS, the population can be categorized into mutually exclusive groups. For instance, it can be divided into categories such as unemployed, employed, and those outside the workforce, with the sum of these categories constituting the total population.

4.1 One block

We assume here that the LFS survey has only one block, denoted $B = 1$. The observations in this block can be divided into J waves. Let y_t^j $(j = 1, 2, \dots, J)$ be the estimate of an LFS figure (measured relative to the LFS population) based on the observations in wave j at time t.

Each wave estimate y_t^j is decomposed into the overall estimate of the LFS figure at time t, denoted μ_t, a wave-specific effect λ_t^j, which may vary over time, and a wave-specific survey error u_t^j.

$$y_t^j = \mu_t + \lambda_t^j + u_t^j \text{ for } t = 1, 2, \dots, T. \tag{1}$$

To identify the wave-specific effects, we impose a restriction on them. The general restriction is formulated as

$$\sum_{j=1}^{J} \omega^j \lambda_t^j = 0 \text{ for all } t \text{ with } \sum_{j=1}^{J} \omega^j = 1, \tag{2}$$

where the weights ω^j are fixed. Typically, the wave-specific effects are assumed to have a mean of zero, so the weights are set to $\omega^j = \frac{1}{J}$ for all j. However, alternative identification restrictions may be applied. For example, in the LFS of the Netherlands, the first wave is assumed to be unbiased, so the weights are specified as $\omega^1 = 1$ and $\omega^j = 0$ for $j = 2, \dots, J$.

The rotating panel design of the LFS introduces correlations between observations in one wave at time t and those from the previous wave at time $t - l$, where l is the number of periods between re-interviews. This correlation is absent for individuals in the first wave, as they are interviewed for the first time in period t. The error process for each wave is given by

$$\begin{aligned} & u_t^1 = \varepsilon_t^1 \text{ for } t = 1, 2, \dots, T \\ & u_t^j = \begin{cases} \varepsilon_t^j & t = 1, 2, \dots, l \\ \phi u_{t-l}^{j-1} + \sqrt{1-\phi^2}\varepsilon_t^j & t = l+1, l+2, \dots, T \end{cases} \quad j = 2, 3, \dots, J \\ & \quad \text{with } 0 \leq \phi < 0,\ \varepsilon_t^j \sim N(0, \sigma_\varepsilon^2) \text{ for } t = 1, 2, \dots, T \text{ and } j = 1, 2, \dots, J, \end{aligned} \tag{3}$$

where ϕ is the autoregressive parameter. The unconditional variance of u_t^j is equal across waves and equal to σ_ε^2. This is derived by taking the variance of both sides of the equation for u_t^j in (3).

By combining (1) and (3), we obtain

$$y_t^j = \mu_t + \lambda_t^j + \phi \left[y_{t-l}^{j-1} - \mu_{t-l} - \lambda_{t-l}^{j-1} \right] 1_{t>l \& j>1} + \sqrt{1 - \phi^2 1_{j>1}} \varepsilon_t^j, \quad (4)$$

where 1_a denotes the indicator function taking the value of 1 if a is true, and 0 otherwise.

The composite estimator, derived by minimizing the residual variance across waves $S_t = \sum_{j=1}^{J} (\varepsilon_t^j)^2$ with respect to μ_t (for $t > l$), is given by

$$\hat{\mu}_t = (1 - \tilde{\alpha}) \hat{\mu}_t^{dir} + \tilde{\alpha} \left[\hat{\mu}_{t-l} + \frac{1}{J-1} \sum_{j=2}^{J} \left(y_t^j - y_{t-l}^{j-1} \right) \right] + \tilde{\beta} \frac{1}{J} \left[y_t^1 - \frac{1}{J-1} \sum_{j=2}^{J} y_t^j \right] + \tilde{\gamma}_t, \quad (5)$$

where

$$\hat{\mu}_t^{dir} = \frac{1}{J} \sum_{i=1}^{J} y_t^j. \quad (6)$$

The the estimates of the parameters α, and β are defined as

$$\tilde{\alpha} = \tilde{\phi} \frac{J-1}{J - \tilde{\phi}^2}, \quad (7)$$

$$\tilde{\beta} = \tilde{\phi} \frac{(J-1)(1-\tilde{\phi})}{J - \tilde{\phi}^2} = (1 - \tilde{\phi}) \tilde{\alpha}, \quad (8)$$

where $\tilde{\phi}$ is some estimate of ϕ. For these and other estimates, we use a tilde to indicate that they are not necessarily based on minimizing of S_t. Finally, the estimate of γ_t is a function of wave-specific effects that are allowed to be time-varying. In (5), α captures the relative weight of the composite estimate from l months ago, β adjusts for deviations in wave-specific effects, and γ_t accounts for any residual wave-specific bias. If the wave-specific effects are time-invariant, and the weights are equal such that $\sum_{j=1}^{J} \lambda^j = 0$, we have $\tilde{\gamma} = \frac{\tilde{\phi}}{J - \tilde{\phi}^2} \left(\tilde{\phi} \tilde{\lambda}^1 - \tilde{\lambda}^J \right)$ where $\tilde{\lambda}^j$ is some estimate of λ^j.

Note that the composite estimator in (5) is a weighted average of two components. The first component is the direct estimate of the LFS figure, $\hat{\mu}_t^{dir}$, and the second component is the composite estimate from l months ago, adjusted for changes in the sample overlap between periods. In addition, the estimator depends on the correction between the first wave and the other waves, as well as a term that captures the effects of the wave-specific effects.

To refine the composite estimator, we introduce the bias-corrected estimate $y_t(\lambda)^j = y_t^j - \tilde{\lambda}_t^j$.

The estimator can then be reformulated as

$$\begin{aligned}\hat{\mu}_t =&(1-\tilde{\alpha})\hat{\mu}(\lambda)_t^{dir} \\ &+\tilde{\alpha}\left[\hat{\mu}_{t-l}+\frac{1}{J-1}\sum_{j=2}^{J}\left(y(\lambda)_t^j - y(\lambda)_{t-l}^{j-1}\right)\right] \\ &+\tilde{\beta}\frac{1}{J}\left[y(\lambda)_t^1 - \frac{1}{J-1}\sum_{j=2}^{J} y(\lambda)_t^j\right] \text{ for } t = l+1, l+2, \ldots, T,\end{aligned} \tag{9}$$

where $\hat{\mu}(\lambda)_t^{dir} = \frac{1}{J}\sum_{j=1}^{J} y(\lambda)_t^j$. Assuming that the average of the wave-specific effects is zero, i.e., $\omega^j = 1/J$, we have $\hat{\mu}(\lambda)_t^{dir} = \hat{\mu}_t^{dir}$ (as $\hat{\mu}(\lambda)_t^{dir} = \frac{1}{J}\sum_{j=1}^{J} y_t^j - \frac{1}{J}\sum_{j=1}^{J}\tilde{\lambda}_t^j = \frac{1}{J}\sum_{j=1}^{J} y_t^j$). For initial observations ($t = 1, 2, \ldots, l$), the composite estimator is simply the direct estimate:

$$\hat{\mu}_t = \hat{\mu}(\lambda)_t^{dir} \text{ for } t = 1, 2, \ldots, l. \tag{10}$$

Remark 4.1 *The last term in (9) can be reformulated as*

$$\tilde{\beta}\frac{1}{J}\left[y(\lambda)_t^1 - \frac{1}{J-1}\sum_{j=2}^{J} y(\lambda)_t^j\right] = \tilde{\beta}\frac{1}{J}\left[\frac{J}{J-1}y(\lambda)_t^1 - \frac{1}{J-1}\sum_{j=1}^{J} y(\lambda)_t^j\right] = \tilde{\beta}\frac{1}{J-1}\left[y(\lambda)_t^1 - \hat{\mu}(\lambda)_t^{dir}\right],$$

which shows that this term adjusts for how the bias-corrected first-wave observation deviates from the average.

Next, we reformulate the second term in (9) as

$$\begin{aligned}&\tilde{\alpha}\left[\hat{\mu}_{t-l}+\frac{1}{J-1}\sum_{j=2}^{J}\left(y(\lambda)_t^j - y(\lambda)_{t-l}^{j-1}\right)\right] \\ =&\tilde{\alpha}\left[\hat{\mu}_{t-l}+\left(\hat{\mu}(\lambda)_t^{dir}-\hat{\mu}(\lambda)_{t-l}^{dir}\right)+\frac{1}{J-1}\left[\left(y(\lambda)_{t-l}^J-\hat{\mu}(\lambda)_{t-l}^{dir}\right)-\left(y(\lambda)_t^1-\hat{\mu}(\lambda)_t^{dir}\right)\right]\right].\end{aligned}$$

Using $\tilde{\beta} = (1-\tilde{\phi})\tilde{\alpha}$, the two last terms in (9) can be combined such that

$$\begin{aligned}\hat{\mu}_t =&(1-\tilde{\alpha})\hat{\mu}(\lambda)_t^{dir} \\ &+\tilde{\alpha}\left[\hat{\mu}_{t-l}+\left(\hat{\mu}(\lambda)_t^{dir}-\hat{\mu}(\lambda)_{t-l}^{dir}\right)+\frac{1}{J-1}\left[\left(y(\lambda)_{t-l}^J-\hat{\mu}(\lambda)_{t-l}^{dir}\right)-\tilde{\phi}\left(y(\lambda)_t^1-\hat{\mu}(\lambda)_t^{dir}\right)\right]\right],\end{aligned}$$

consists of two terms.

Finally, combining all terms yields

$$\begin{aligned}\hat{\mu}_t =&\hat{\mu}(\lambda)_t^{dir} \\ &+\tilde{\alpha}\left[\left(\hat{\mu}_{t-l}-\hat{\mu}(\lambda)_{t-l}^{dir}\right)+\frac{1}{J-1}\left[\left(y(\lambda)_{t-l}^J-\hat{\mu}(\lambda)_{t-l}^{dir}\right)-\tilde{\phi}\left(y(\lambda)_t^1-\hat{\mu}(\lambda)_t^{dir}\right)\right]\right],\end{aligned}$$

where the last term expresses the difference between the composite estimator and the direct esti-

mator. This difference consists of three terms; one expresses the difference between the composite estimator and the direct estimator in period $t-l$; one depends on how wave J deviates from the average across waves in period $t-l$; and one depends on how wave 1 deviates from the average in period t, with a weight depending on the estimate of the correlation parameter ϕ.

4.2 Multiple blocks

Consider the case where there are $B > 1$ blocks, where block b consists of J^b waves ($b = 1, \ldots, B$). A composite estimator can be generated for each block, and these block-specific estimators can be averaged to form an overall composite estimator. The weights in the average are based on the number of waves in each block:

$$\hat{\mu}_t = \sum_{b=1}^{B} \frac{J^b}{J^{tot}} \hat{\mu}_t^{(b)}, \tag{11}$$

where $J^{tot} = \sum_{b=1}^{B} J^b$ and $\hat{\mu}_t^{(b)}$ is the estimate of the LFS variable derived from block b. Equation (11) assumes that the autocorrelation between blocks is negligible.

If the number of waves is the same across all blocks, all the blocks can be combined into one block by averaging the wave-specific estimates across blocks. That is, $y_t^j = \frac{1}{B} \sum_{b=1}^{B} y_t^{j,b}$, where $y_t^{j,b}$ is the estimate based on wave j in block b.

In the U.S. version of the LFS, the Current Population Survey (CPS), a composite estimator similar to (5) is employed to compute various figures, as descibed in U.S. Census Bureau (2019). In the CPS, the parameter $\gamma = 0$ for both unemployment and employment figures. For the unemployment estimates, $\alpha = 0.4$ and $\beta \frac{1}{J} = 0.3$ are used. This value of α corresponds approximately to $\phi = 0.5$ when $J = 4$, but this ϕ value, according to (8), implies that $\beta \frac{1}{J} = 0.05$, which is substantially lower than the value used in the CPS. For employment figures, $\alpha = 0.7$ and $\beta \frac{1}{J} = 0.4$ are used, where $\alpha = 0.7$ roughly corresponds to $\phi = 0.8$. However, this value of ϕ suggests that $\beta \frac{1}{J} = 0.14$, less than half of the value used in the CPS.

Notably, the values of α and $\beta \frac{1}{J}$ in the CPS are not derived from the model above. According to U.S. Census Bureau (2019, p. 77), these values were selected after extensive review of multiple series to minimize the variance of month-to-month changes in unemployment and employment estimates (see also Gurney and Daly, 1965; Breau and Ernst, 1983). Additionally, the U.S. CPS does not explicitly account for wave-specific effects, which may explain the higher values of $\beta \frac{1}{J}$ used in the CPS.

Finally, the number of people not in the labour force is computed residually in the CPS.

5 The multivariate composite estimator

In this section, we extend the composite estimator to an estimator of multiple labour market categories. Consider an LFS population divided into I mutually exclusive categories. For example, if the categories are unemployed, employed, and those outside the workforce, then $I = 3$. The LFS figures in each category are measured as fractions of the total population. Therefore, the sum across all categories equals 1. Consequently, we can express the last category as a function of the others, reducing the problem to a system of $I - 1$ categories.

Define the following vectors for $j = 1, 2, \ldots, J$:

$$\boldsymbol{y}_t^j = \begin{pmatrix} y_t^{1,j} \\ \vdots \\ y_t^{I-1,j} \end{pmatrix}, \boldsymbol{\mu}_t = \begin{pmatrix} \mu_t^1 \\ \vdots \\ \mu_t^{I-1} \end{pmatrix}, \boldsymbol{\lambda}_t^j = \begin{pmatrix} \lambda_t^{1,j} \\ \vdots \\ \lambda_t^{I-1,j} \end{pmatrix}, \boldsymbol{u}_t^j = \begin{pmatrix} u_t^{1,j} \\ \vdots \\ u_t^{I-1,j} \end{pmatrix}, \boldsymbol{\varepsilon}_t^j = \begin{pmatrix} \varepsilon_t^{1,j} \\ \vdots \\ \varepsilon_t^{I-1,j} \end{pmatrix}, \tag{12}$$

where $\boldsymbol{y}_t^j$ is the vector of observations for the j'th wave, and the other vectors represent the corresponding parameters and errors for each wave.

The values for the last category can be calculated residually since they sum to either 0 or 1 across categories; $y_t^{I,j} = 1 - \sum_{i=1}^{I-1} y_t^{i,j}$, $\lambda^{I,j} = -\sum_{i=1}^{I-1} \lambda^{i,j}$, $u_t^{I,j} = -\sum_{i=1}^{I-1} u_t^{i,j}$ $(j = 1, \ldots, J)$, and $\mu_t^I = 1 - \sum_{i=1}^{I-1} \mu_t^i$.

The multivariate generalization of the univariate model in (1) and (3) is given by

$$\boldsymbol{y}_t^j = \boldsymbol{\mu}_t + \boldsymbol{\lambda}_t^j + \boldsymbol{u}_t^j \text{ for } t = 1, 2, \ldots, T \tag{13}$$

$$\boldsymbol{u}_t^j = \begin{cases} \boldsymbol{\varepsilon}_t^j & t = 1, 2, \ldots, l \text{ or } j = 1 \\ \boldsymbol{\Phi} \boldsymbol{u}_{t-l}^{j-1} + \boldsymbol{K} \boldsymbol{\varepsilon}_t^j & t = l+1, l+2, \ldots, T \text{ and } j = 2, 3, \ldots, J, \end{cases} \tag{14}$$

where $\boldsymbol{\Phi}$ is the $(I-1) \times (I-1)$ matrix of autoregressive coefficients. The noise vector, $\boldsymbol{\varepsilon}_t^j$, is Gaussian with variance matrix $\boldsymbol{\Omega}$. For $j > 1$, $\boldsymbol{K}$ is implicitly determined by the variance-covariance structure of $\boldsymbol{u}_t^j$ and $\boldsymbol{\varepsilon}_t^j$, which is described by the equation $\boldsymbol{\Omega} = \boldsymbol{\Phi}\boldsymbol{\Omega}\boldsymbol{\Phi}^\top + \boldsymbol{K}\boldsymbol{\Omega}\boldsymbol{K}^\top$, such that $\boldsymbol{\Omega}$ is the variance-covariance matrix of both $\boldsymbol{u}_t^j$ and $\boldsymbol{\varepsilon}_t^j$ and does not vary with j, and $\top$ is the transpose symbol.

The composite estimator is obtained by minimizing the sum of squared errors, $S_t = \sum_{j=1}^J \varepsilon_t^j \varepsilon_t^{j\top}$, as detailed in Appendix A. This yields the following multivariate composite estimator

$$\begin{aligned} \widehat{\boldsymbol{\mu}}_t = & (\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\alpha}}) \, \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} \\ & + \widetilde{\boldsymbol{\alpha}} \left[\widehat{\boldsymbol{\mu}}_{t-l} + \frac{1}{J-1} \sum_{j=2}^{J} \left(\boldsymbol{y}(\lambda)_t^j - \boldsymbol{y}(\lambda)_{t-l}^{j-1} \right) \right] \\ & + \widetilde{\boldsymbol{\beta}} \frac{1}{J-1} \left[\boldsymbol{y}(\lambda)_t^1 - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} \right] \text{ for } t = l+1, l+2, \ldots, \end{aligned} \tag{15}$$

where $\boldsymbol{y}(\lambda)_t^j = \boldsymbol{y}_t^j - \widetilde{\boldsymbol{\lambda}}_t^j$. The term $\widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}$ is the direct estimator, defined as

$$\widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} = \frac{1}{J} \sum_{j=1}^{J} \boldsymbol{y}(\lambda)_t^j. \tag{16}$$

Additionally, by using $\boldsymbol{K\Omega K}^\top = \boldsymbol{\Omega} - \boldsymbol{\Phi\Omega\Phi}^\top$, the estimate of the parameter matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are given by

$$\widetilde{\boldsymbol{\alpha}} = \left[J\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\right]^{-1} \left[(J-1)\,\widetilde{\boldsymbol{\Phi}}\right] \tag{17}$$

$$\begin{aligned} \widetilde{\boldsymbol{\beta}} &= \left[J\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\right]^{-1} \left[(J-1)\,\widetilde{\boldsymbol{\Phi}}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\right)\right] \\ &= \widetilde{\boldsymbol{\alpha}}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\right) \end{aligned} \tag{18}$$

where $\boldsymbol{I}_n$ being the identity matrix of dimension n; see Appendix A.

As for the univariate model, we have

$$\widehat{\boldsymbol{\mu}}_t = \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} \text{ for } t = 1, 2, \ldots, l. \tag{19}$$

If the survey consists of multiple blocks, a multivariate version of (11) can be applied.

Remark 5.1 *Using $\widetilde{\boldsymbol{\beta}} = \widetilde{\boldsymbol{\alpha}}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\right)$, the two last terms in (15) can be rewritten as*

$$\widetilde{\boldsymbol{\alpha}}\left[\widehat{\boldsymbol{\mu}}_{t-l} + \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} - \widehat{\boldsymbol{\mu}}(\lambda)_{t-l}^{dir} + \frac{1}{J-1}\left[\left(\boldsymbol{y}(\lambda)_{t-l}^J - \widehat{\boldsymbol{\mu}}(\lambda)_{t-l}^{dir}\right) - \widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\left(\boldsymbol{y}(\lambda)_t^1 - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right)\right]\right].$$

Combining terms yields a vector autoregressive process of order 1 of the difference between the multivariate composite estimate and the direct estimate, given by

$$\left[\widehat{\boldsymbol{\mu}}_t - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right] = \widetilde{\boldsymbol{\alpha}}\left[\widehat{\boldsymbol{\mu}}_{t-l} - \widehat{\boldsymbol{\mu}}(\lambda)_{t-l}^{dir}\right] + \widetilde{\boldsymbol{\xi}}_t, \tag{20}$$

where the vector

$$\widetilde{\boldsymbol{\xi}}_t = \widetilde{\boldsymbol{\alpha}}\frac{1}{J-1}\left[\left(\boldsymbol{y}(\lambda)_{t-l}^J - \widehat{\boldsymbol{\mu}}(\lambda)_{t-l}^{dir}\right) - \widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\left(\boldsymbol{y}(\lambda)_t^1 - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right)\right]$$

captures innovation effects, representing deviations of wave 1 and wave J from the direct estimates in periods t and $t-l$, respectively.

6 Estimates of the parameters

We need estimates of the coefficient matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as well as the coefficient vector $\boldsymbol{\lambda}_t^j$ (for all j) to apply (15) to get the composite estimates. The estimates of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ will depend on the estimates of $\boldsymbol{\Phi}$ and $\boldsymbol{\Omega}$. In this section, we derive pseudo panel-survey errors in Section 6.1 and apply these to obtain estimates of $\mathbf{\Phi}$ and $\mathbf{\Omega}$ in Section 6.2.

If the wave-specific effects, $\boldsymbol{\lambda}_t^j$, are time-invariant, the estimates of these are provided in Section 6.1. The estimation of $\boldsymbol{\lambda}_t^j$ when it is allowed to vary over time, is considered in Section 7. In that case, the estimates of $\mathbf{\Phi}$ and $\mathbf{\Omega}$ can still be obtained by following the approach outlined in Section 6.2.

6.1 Estimates of time-invariant wave-specific effects

Here, we outline the procedure for estimating the time-invariant wave-specific effects, $\boldsymbol{\lambda}^j$. To proceed with the estimation, we apply the concept of pseudo panel-survey errors, which provides preliminary insights into the survey errors (see Pfeffermann et al., 1998). These pseudo-errors are applied to estimate the parameter matrices such as $\mathbf{\Phi}$ and $\mathbf{\Omega}$.

The vector of pseudo-errors, denoted $\tilde{\boldsymbol{u}}_t^j$, is defined as

$$\tilde{\boldsymbol{u}}_t^j = \boldsymbol{y}_t^j - \tilde{\boldsymbol{\mu}}_t - \widetilde{\boldsymbol{\lambda}}^j. \tag{21}$$

Here, $\tilde{\boldsymbol{\mu}}_t$ represent initial estimates of the overall mean, obtained through averages. Specifically, the estimates can be given by $\tilde{\boldsymbol{\mu}}_t = \overline{\boldsymbol{y}}_t$ and $\widetilde{\boldsymbol{\lambda}}^j = \overline{\boldsymbol{y}}^j - \overline{\boldsymbol{y}}$; where $\overline{\boldsymbol{y}}_t$, $\overline{\boldsymbol{y}}^j$, and $\overline{\boldsymbol{y}}$ are averages accross the J waves, the T observations, and both, respectively.

Alternatively, weighted averages can be used when the weights in (2) are assigned to each wave j $(j = 1, 2, \ldots, J)$. In this case, the estimates are given by

$$\tilde{\boldsymbol{\mu}}_t = \overline{\boldsymbol{y}}_t^{(\omega)}, \quad t = 1, 2, \ldots, T \tag{22}$$

$$\widetilde{\boldsymbol{\lambda}}^j = \overline{\boldsymbol{y}}^j - \overline{\boldsymbol{y}}^{(\omega)}, \quad j = 1, 2, \ldots, J \tag{23}$$

where

$$\overline{\boldsymbol{y}}_t^{(\omega)} = \sum_{j=1}^{J} \omega^j \boldsymbol{y}_t^j, \quad \overline{\boldsymbol{y}}^j = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{y}_t^j, \text{ and } \quad \overline{\boldsymbol{y}}^{(\omega)} = \frac{1}{T}\sum_{t=1}^{T}\sum_{j=1}^{J} \omega^j \boldsymbol{y}_t^j.$$

The weighted averages provide more accurate preliminary estimates of $\boldsymbol{\lambda}^j$ since the weighted averages correspond more closely to the underlying definitions. For instance, if we assume that the first wave is unbiased by setting $\omega^1 = 1$ and $\omega^j = 0$ for $j = 2, \ldots, J$, the preliminary estimate of the wave-specific effect for wave 1 will be zero: $\tilde{\boldsymbol{\lambda}}^1 = \overline{\boldsymbol{y}}^1 - \overline{\boldsymbol{y}}^{(\omega)} = \overline{\boldsymbol{y}}^1 - \overline{\boldsymbol{y}}^1 = 0$.

6.2 Autocovariance and autocorrelation matrices

In this section, we derive the estimates of the autocorrelation matrix $\mathbf{\Phi}$, which is key to understanding the relationships between variables at different time lags. We also provide the estimation procedure for the variance matrix $\mathbf{\Omega}$, which reflects the model's overall variability. To do so, we need to define the estimates of the autocovariance matrices $\mathbf{\Gamma}_0$ and $\mathbf{\Gamma}_1$.

$$\widetilde{\mathbf{\Gamma}}_0 = \frac{1}{T}\frac{1}{J}\sum_{t=1}^{T}\sum_{j=1}^{J}\tilde{\boldsymbol{u}}_t^j\tilde{\boldsymbol{u}}_t^{j\top} \text{ and } \widetilde{\mathbf{\Gamma}}_1 = \frac{1}{T-l}\frac{1}{J-1}\sum_{t=1}^{T-l}\sum_{j=1}^{J-1}\tilde{\boldsymbol{u}}_{t+l}^{j+1}\tilde{\boldsymbol{u}}_t^{j\top}$$

We apply the Yule-Walker equation to derive the estimate of $\mathbf{\Phi}$. The univariate version was described in Yule (1907), and the multivariate approach was derived in Whittle (1963). The multivariate Yule-Walker equation imply $\widetilde{\mathbf{\Gamma}}_1 = \widetilde{\mathbf{\Phi}}\widetilde{\mathbf{\Gamma}}_0$. Therefore, using the Yule-Walker equation, we derive an estimator for $\mathbf{\Phi}$ as follows:

$$\widetilde{\mathbf{\Phi}} = \widetilde{\mathbf{\Gamma}}_1\widetilde{\mathbf{\Gamma}}_0^{-1}. \tag{24}$$

Finally, the variance matrix $\mathbf{\Omega}$ is estimated as the autocovariance matrix at lag 0, reflecting the model's overall variance structure:

$$\widetilde{\mathbf{\Omega}} = \widetilde{\mathbf{\Gamma}}_0. \tag{25}$$

7 Time-varying wave-specific effects

In this section, we derive a simple formula for updating the estimates of the wave-specific effects in case they are time-varying. To do so, we first need to remove the dependencies between the wave-specific effects, which is done in Section 7.1. In Section 7.2, we estimate the signal-to-noise ratio and derive the updating formula based on this estimate.

7.1 Interdependence of time-varying wave-specific effects

In Section 6.1, we assumed that the wave-specific effects for each category were time-invariant. However, this assumption may be overly restrictive, cf. Krueger et al. (2017). To address this, we explore models that incorporate time-varying wave-specific effects, building on the approaches of van den Brakel and Krieg (2009) and Hungnes et al. (2024). These models allow for greater flexibility and more accurate representations of survey error structures.

van den Brakel and Krieg (2009) assume that the first wave of each category is unbiased, i.e. $\lambda_t^{i,1} = 0$ for all t and i. Furthermore, they assume that subsequent waves follow random-walk processes.

Elliott and Zong (2019) further develop this for the case where it is assumed that the sum of the wave-specific effects is zero, i.e. $\sum_{j=1}^{J}\lambda_t^{i,j} = 0$ for all t. They then assume that the wave-specific effect of one of the waves (e.g., the last) is given as the negative sum of the others, while the remaining wave-specific effects follow independent random-walk processes.

Hungnes et al. (2024) further argue that this framework leads to greater variance in the wave-specific effect of the residually determined wave. To address this, they propose a specification where the innovations in the wave-specific effects are negatively correlated. They also consider the case where the sum of wave-specific effects is zero, implying that the weights ω^j are equal, i.e., $\omega^j = \frac{1}{J}$ for all j.

There is a dependency between the J wave-specific effects through the one restriction in (2) between these effects. This implies that $J-1$ independent processes can describe the wave-specific effects. We define the matrix $\boldsymbol{W}$ to derive these independent processes. In the definition below, this matrix is given as orthogonal to an arbitrary vector $\boldsymbol{v}$; and in Remark 7.1, we show how it can be derived:

Definition 7.1 *Let $\boldsymbol{v}$ be a column vector of dimension n. Define $\boldsymbol{W}^{\boldsymbol{v}}$ as an $n\times(n-1)$ matrix whose columns are orthogonal to $\boldsymbol{v}$, i.e., $(\boldsymbol{W}^{\boldsymbol{v}})^\top \boldsymbol{v} = \boldsymbol{0}_{n-1}$, and normalized such that $(\boldsymbol{W}^{\boldsymbol{v}})^\top \boldsymbol{W}^{\boldsymbol{v}} = \boldsymbol{I}_{n-1}$.*

Remark 7.1 *We can partition $\boldsymbol{v}$ into two components: the first element, $\boldsymbol{v}_1$, and the remaining elements, v_{-1}, so that $\boldsymbol{v} = \left(\boldsymbol{v}_1, v_{-1}^\top\right)^\top$. If v_1 is non-zero, an orthogonal matrix to $\boldsymbol{v}$ can be constructed as*

$$\boldsymbol{W}^{*\boldsymbol{v}} = \begin{pmatrix} -(v_1)^{-1} \cdot \boldsymbol{v}_{-1}^\top \\ \boldsymbol{I}_{n-1} \end{pmatrix}.$$

This matrix can be normalized by

$$\boldsymbol{W}^{\boldsymbol{v}} = \boldsymbol{W}^{*\boldsymbol{v}} \left((\boldsymbol{W}^{*\boldsymbol{v}})^\top \boldsymbol{W}^{*\boldsymbol{v}}\right)^{-1/2},$$

as this ensures $(\boldsymbol{W}^{\boldsymbol{v}})^\top \boldsymbol{W}^{\boldsymbol{v}} = \boldsymbol{I}_{n-1}$.

Note also that $\boldsymbol{S} = (\boldsymbol{W}^{*\boldsymbol{v}})^\top \boldsymbol{W}^{*\boldsymbol{v}}$ is both symmetric and positive definite. A symmetric matrix can be decomposed using an eigen-decomposition as $\boldsymbol{S} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^\top$, where $\boldsymbol{\Lambda}$ is a diagonal matrix holding the eigenvalues and $\boldsymbol{V}$ a matrix with the corresponding eigenvectors. As $\boldsymbol{S}$ is also positive definite, we have $\boldsymbol{S}^n = \boldsymbol{V}\boldsymbol{\Lambda}^n\boldsymbol{V}^\top$ when n is a real number. Here, we apply this for $n = -1/2$.

We can now define the process for $J-1$ wave-specific effects in the absence of interdependence across waves as

$$\boldsymbol{\lambda}_t^{i,*} = \boldsymbol{\lambda}_{t-1}^{i,*} + \boldsymbol{\eta}_t^{i,*},\ \boldsymbol{\eta}_t^{i,*} \sim N\left(\boldsymbol{0}_{J-1}, \boldsymbol{I}_{J-1}\sigma_\lambda^2\right). \tag{26}$$

Pre-multiplying both sides of (26) by $\boldsymbol{W}^{\boldsymbol{\omega}}$, an orthogonal matrix to the vector of weights $\boldsymbol{\omega} = (\omega^1, \omega^2, \ldots, \omega^J)^\top$, and using $\boldsymbol{\lambda}_t^i = \boldsymbol{W}^{\boldsymbol{\omega}}\boldsymbol{\lambda}_t^{i,*}$ and $\boldsymbol{\eta}_t^i = \boldsymbol{W}^{\boldsymbol{\omega}}\boldsymbol{\eta}_t^{i,*}$, yields

$$\boldsymbol{\lambda}_t^i = \boldsymbol{\lambda}_{t-1}^i + \boldsymbol{\eta}_t^i,\ \boldsymbol{\eta}_t^i \sim N\left(\boldsymbol{0}_J, \left(\boldsymbol{I}_J - \boldsymbol{\omega}\left(\boldsymbol{\omega}^\top\boldsymbol{\omega}\right)^{-1}\boldsymbol{\omega}^\top\right)\sigma_\lambda^2\right),\ \boldsymbol{\omega}^\top\boldsymbol{\lambda}_0^i = 0, \tag{27}$$

where the covariance structure reflects the residual variability in the wave-specific effects after accounting for the weighted sum constraint, ensuring that the effects sum to zero under the given weighting scheme, i.e. $\boldsymbol{\omega}^\top\boldsymbol{\lambda}_t^i = 0$ given that this condition is met in period 0.

Equation (27) models the evolution of wave-specific effects under the weighting scheme given by (2), capturing both interdependence across waves and the constraints imposed by the weights. This framework generalizes earlier approaches by van den Brakel and Krieg (2009) and Hungnes et al. (2024), allowing for more flexible modeling of wave-specific effects in survey data. For a more compact notation, we define $\boldsymbol{a}_b$ as a column vector of b a's.

Remark 7.2 *The process in (27) can recover the specific cases of earlier models:*

- *For equal weights, $\boldsymbol{\omega} = \frac{1}{J}\mathbf{1}_J$, the term for the covariance matrix, given by $\boldsymbol{I}_J - \boldsymbol{\omega}\left(\boldsymbol{\omega}^\top\boldsymbol{\omega}\right)^{-1}\boldsymbol{\omega}^\top$, reduces to $\boldsymbol{I}_J - \frac{1}{J}\mathbf{1}_J\mathbf{1}_J^\top$, as in the model by Hungnes et al. (2024).*
- *If we assume that the first wave is unbiased, $\boldsymbol{\omega}^\top = \left(1, \mathbf{0}_{J-1}^\top\right)$ as in van den Brakel and Krieg (2009), we get the covariance*

$$\boldsymbol{I}_J - \boldsymbol{\omega}\left(\boldsymbol{\omega}^\top\boldsymbol{\omega}\right)^{-1}\boldsymbol{\omega}^\top = \boldsymbol{I}_J - \begin{pmatrix} 1 & \mathbf{0}_{J-1}^\top \\ \mathbf{0}_{J-1} & \mathbf{0}_{J-1}\mathbf{0}_{J-1}^\top \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0}_{J-1}^\top \\ \mathbf{0}_{J-1} & \boldsymbol{I}_{J-1} \end{pmatrix}$$

combined with the restriction $\lambda_0^{i,1} = 0$ for the initial values. This implies that $\lambda_t^{i,1} = 0$ for all t and the wave-specific effects for the remaining $J-1$ waves follow independent random walks as in (26).

It is also important to account for the fact that within each wave, the wave-specific effects must sum to zero across categories, i.e., $\sum_{i=1}^{I}\lambda_t^{i,j} = 0$ for all j. This implies, e.g. that if unemployment is usually measured too high in wave j, those employed and/or those outside the labour market must be measured too low in the same wave (since all must be assigned to one of the mutually disjunctive categories). This means that we also have a dependency between the LFS categories for the wave-specific effects within wave j. Equation (28) captures this dependency structure, ensuring consistency across categories.

$$\begin{pmatrix} \lambda_t^{1,j} \\ \lambda_t^{2,j} \\ \vdots \\ \lambda_t^{I,j} \end{pmatrix} = \begin{pmatrix} \lambda_{t-1}^{1,j} \\ \lambda_{t-1}^{2,j} \\ \vdots \\ \lambda_{t-1}^{I,j} \end{pmatrix} + \begin{pmatrix} \eta_t^{1,j} \\ \eta_t^{2,j} \\ \vdots \\ \eta_t^{I,j} \end{pmatrix}, \quad \begin{pmatrix} \eta_t^{1,j} \\ \eta_t^{2,j} \\ \vdots \\ \eta_t^{I,j} \end{pmatrix} \sim N\left(\mathbf{0}_I, \left(\boldsymbol{I}_I - \frac{1}{I}\mathbf{1}_I\mathbf{1}_I^\top\right)\sigma_\lambda^2\right), \quad \sum_{i=1}^{I}\lambda_0^{i,j} = 0. \tag{28}$$

By pre-multiplying both sides of (28) with $(\boldsymbol{W}^{\boldsymbol{v}})^\top$ where $\boldsymbol{v} = \mathbf{1}_I$, we obtain $I-1$ independent random walk processes with equal variance:

$$(\boldsymbol{W}^{\mathbf{1}_I})^\top \left[\begin{pmatrix} \lambda_t^{1,j} \\ \lambda_t^{2,j} \\ \vdots \\ \lambda_t^{I,j} \end{pmatrix} - \begin{pmatrix} \lambda_{t-1}^{1,j} \\ \lambda_{t-1}^{2,j} \\ \vdots \\ \lambda_{t-1}^{I,j} \end{pmatrix}\right] \sim N\left(\mathbf{0}_{I-1}, \boldsymbol{I}_{I-1}\sigma_\lambda^2\right).$$

Hence, in modeling time-varying wave-specific effects, we must consider dependencies both between waves and across categories. One way to deal with this is to study a subset of these wave-specific effects. For each category, we analyze the $J-1$ independent wave effects, as in (26); for each wave, we consider the $I-1$ independent categories, as in (28). This approach collectively examines $(I-1)(J-1)$ independent processes, each of which follows an independent random walk.

Let $\boldsymbol{\lambda}_t^{dep}$ represent the vector of all $I \cdot J$ dependent wave-specific effects across categories and waves:

$$\boldsymbol{\lambda}_t^{dep} = \begin{pmatrix} (\boldsymbol{\lambda}^{dep})_t^1 \\ (\boldsymbol{\lambda}^{dep})_t^2 \\ \vdots \\ (\boldsymbol{\lambda}^{dep})_t^I \end{pmatrix}, \text{ where } (\boldsymbol{\lambda}^{dep})_t^i = \begin{pmatrix} \lambda_t^{i,1} \\ \lambda_t^{i,2} \\ \vdots \\ \lambda_t^{i,J} \end{pmatrix} \text{ for } i = 1, 2, \dots, I,$$

represents the wave-specific effects for category i.

The $(I-1)(J-1)$ independent processes can be derived by pre-multiplying $\boldsymbol{\lambda}_t^{dep}$ by:

$$(\boldsymbol{W}^{\boldsymbol{1}_I})^\top \otimes (\boldsymbol{W}^{\boldsymbol{w}})^\top = (\boldsymbol{W}^{\boldsymbol{1}_I} \otimes \boldsymbol{W}^{\boldsymbol{w}})^\top,$$

where $\otimes$ indicates the Kronecker product.

7.2 State-space formulation and application of the Kalman filter for wave-specific effects

In this section, we derive a simple formula for updating the time-varying wave-specific effects. Accurately updating time-varying wave-specific effects is essential for maintaining reliable population-level estimates, especially if there are substantial changes in them. By formulating the model in state-space form, we derive a steady-state updating rule where the weights, independent of time, are governed by the signal-to-noise ratio.

Let $\boldsymbol{x}_t^{dep}$ be the vector of dependent observed deviations from the preliminary estimate of the population level for each wave and each category at time t, given as

$$\boldsymbol{x}_t^{dep} = \begin{pmatrix} \boldsymbol{y}_t^1 - \tilde{\mu}_t^1 \boldsymbol{1}_J \\ \boldsymbol{y}_t^2 - \tilde{\mu}_t^2 \boldsymbol{1}_J \\ \vdots \\ \boldsymbol{y}_t^I - \tilde{\mu}_t^I \boldsymbol{1}_J, \end{pmatrix}$$

where $\tilde{\mu}_t^i$ is some estimate of μ_t^i, for example $\tilde{\mu}_t^i = J^{-1} \sum_{j=1}^J y_t^{i,j}$. Let $\boldsymbol{x}_t^{ind}$ represent the vector of independent observations of $\boldsymbol{x}_t^{dep}$;

$$\boldsymbol{x}_t^{ind} = (\boldsymbol{W}^{\boldsymbol{1}_I} \otimes \boldsymbol{W}^{\boldsymbol{w}})^\top \boldsymbol{x}_t^{dep}.$$

This transformation isolates independent components of the observed deviations for subsequent analysis.

We decompose this vector of the $(I-1)(J-1)$ independent observations of $\boldsymbol{x}_t^{dep}$ into a permanent and an irregular component. This decomposition separates the permanent wave-specific effects, modelled as random walks, from irregular noise components, enabling a more robust estimation of trends over time. The permanent component corresponds to the $(I-1)(J-1)$ independent wave-specific processes we get by pre-multiplying $\boldsymbol{\lambda}_t$ with $(\boldsymbol{W}^{\mathbf{1}_I} \otimes \boldsymbol{W}^{\boldsymbol{w}})^\top$. This vector of independent wave-specific effects is therefore modelled as independent random walks. The random walk assumption reflects the expectation that wave-specific effects evolve gradually over time:

$$\boldsymbol{x}_t^{ind} = \boldsymbol{L}_t + \boldsymbol{V}_t,\ \boldsymbol{V}_t \sim (\mathbf{0}_{(J-1)(I-1)}, \sigma_V^2 \boldsymbol{I}_{(J-1)(I-1)}) \tag{29}$$

$$\boldsymbol{L}_t = \boldsymbol{L}_{t-1} + \boldsymbol{e}_t,\ \boldsymbol{e}_t \sim (\mathbf{0}_{(J-1)(I-1)}, \sigma_L^2 \boldsymbol{I}_{(J-1)(I-1)}), \tag{30}$$

where σ_V^2 and σ_L^2 are the hyperparameters for the irregular component and the level-trend component, respectively. Furthermore, $\boldsymbol{L}_t$ is the vector of the trend-level estimate of the wave-specific effects. The diffuse initial value is given by

$$\boldsymbol{L}_0 \sim (\mathbf{0}, \kappa \boldsymbol{I}_{(J-1)(I-1)}) \tag{31}$$

with $\kappa \to \infty$. Based on this model, we can estimate the signal-to-noise ratio σ_L^2/σ_V^2 on the T first observations ($t = 1, 2, \ldots, T$). We denote this estimate q.

Let l_t be any element of $\boldsymbol{L}_t$. Since the elements are independent, we analyze each element individually. The Kalman filter for this element is given by (see Harvey, 1989, pp. 107–108)

$$l_{t+1|t} = (1 - k_t) l_{t|t-1} + k_t x_t,$$

where x_t corresponds to the element in $\boldsymbol{x}_t^{ind}$ associated with l_t in $\boldsymbol{L}_t$, and the gain, k_t, is

$$k_t = \frac{p_{t|t-1}}{1 + p_{t|t-1}},$$

with the estimate of the variance of $l_{t+1|t}$ in period $t+1$ based on observations up to period t given by the discrete time Riccati difference equation

$$p_{t+1|t} = p_{t|t-1} - \frac{p_{t|t-1}^2}{1 + p_{t|t-1}} + q = \frac{p_{t|t-1}}{1 + p_{t|t-1}} + q.$$

Starting with $p_{1|0} = \kappa$ with $\kappa \to \infty$, which corresponds to a diffuse initial value, we obtain

$$l_{2|1} = x_1, \quad p_{2|1} = 1 + q.$$

The steady state solution occurs when $\lim_{t\to\infty} p_{t+1|t} = \bar{p}$. In this case, the Riccati equation simplifies to (see Harvey, 1989, p. 119)

$$\bar{p} = \frac{\bar{p}}{1+\bar{p}} + q,$$

which leads to the quadratic equation

$$\bar{p}^2 - q\bar{p} - q = 0,$$

with the positive solution

$$\bar{p} = \frac{q + \sqrt{q^2 + 4q}}{2}. \tag{32}$$

Thus, the steady-state Kalman gain is

$$\bar{k} = \frac{\bar{p}}{1+\bar{p}}. \tag{33}$$

The update formula for any element in $\boldsymbol{L}_t$ is then

$$\tilde{l}_t = (1-\bar{k})\tilde{l}_{t-1} + \bar{k}x_t, \;\; t = T+1, T+2, \ldots, \tag{34}$$

where $\tilde{l}_T$ is an estimate for the corresponding element of L_T, where we can use either the estimate provided by the Kalman filter or the smoothed estimate. This weighted combination balances prior estimates and new observations, with $\bar{k}$ controlling the influence of recent data. Since $\bar{k}$ is the same for all elements in $\boldsymbol{L}_t$, we can express the update in vector form as

$$\widetilde{\boldsymbol{L}}_t = (1-\bar{k})\widetilde{\boldsymbol{L}}_{t-1} + \bar{k}\boldsymbol{x}_t^{dep}, \;\; t = T+1, T+2, \ldots. \tag{35}$$

where $\widetilde{\boldsymbol{L}}_t$ is the vector of all estimates in $\boldsymbol{L}_t$.

For updating the wave-specific effects, we use the formula

$$\widetilde{\boldsymbol{\lambda}}_t = \left(\boldsymbol{W}^{\boldsymbol{1}_I} \otimes \boldsymbol{W}^{\boldsymbol{w}}\right)\widetilde{\boldsymbol{L}}_t, \;\; t = T+1, T+2, \ldots. \tag{36}$$

Remark 7.3 *For the widely used special cases where the weights ω are equal or where one wave is unbiased for all categories, (36) combined with (35) can be simplified:*

- *For equal weights, $\boldsymbol{\omega} = \frac{1}{J}\boldsymbol{1}_J$ (see Hungnes et al., 2024), we have*

$$\tilde{\lambda}_t^{i,j} = (1-\bar{k})\tilde{\lambda}_{t-1}^{i,j} + \bar{k}\left(y_t^{i,j} - \tilde{\mu}_t^i\right), \tag{37}$$

  which can be used for updating after period T for $t = T+1, T+2, \ldots,$ $j = 1, 2, \ldots, J,$ and $i = 1, 2, \ldots, I$.

Table 4: Descriptive statistics

	Males, 15–24 y.o.		Males, 25–74 y.o.		Females, 15–24 y.o.		Females, 25-74 y.o.	
	unempl.	empl.	unempl.	empl.	unempl.	empl.	unempl.	empl.
mean*100	5.96	50.97	2.32	75.50	4.88	52.41	1.77	68.73
st.d.*100	1.41	3.47	0.61	1.62	1.35	3.39	0.43	1.14

- *If we assume that the first wave is unbiased,* $\boldsymbol{\omega}^\top = \left(1, \mathbf{0}_{J-1}^\top\right)$ *(see van den Brakel and Krieg, 2009), we have* $\tilde{\lambda}_t^{i,j} = 0$ *for* $i = 1$*, while (37) applies to the remaining waves.*

Remark 7.4 *For relatively small samples, the steady state* $\bar{k}$ *might deviate much from* k_t*. In such cases, one could consider using* k_t *in the updating functions. In particular, if* $\bar{k} < 1/(T+1)$*, where the right-hand side represent the weight of a new observation if the wave-specific effects are time-invariant, one should at least put this weight on the new observation.*

8 Application to the Norwegian LFS

This section applies the multivariate composite estimator to the Norwegian LFS. In Section 8.1, we assume time-invariant wave-specific effects. In Section 8.2, we derive the changes in the LFS figures for employment, unemployment, and non-participants (those outside the labour force) by switching from a direct estimator to the proposed multivariate composite estimator. Section 8.3 investigates whether the wave-specific effects are time-varying or time-invariant.

We use monthly data from the Norwegian Labour Force Survey (LFS) covering the period from January 2006 to December 2020. The start date aligns with that of Hungnes et al. (2024), while the end date reflects a restructuring of the LFS across all Eurostat countries at the turn of the year 2020/2021.

Norway's LFS is characterized by a high response rate. The non-response rate during the study period ranged from 14 to 21 percent (see Eurostat, 2022), with the response rate calculated at the individual level. Among Eurostat countries that compute the individual-level response rate, Norway has the lowest non-response rate. Switzerland has the second-lowest, while many other countries with similar reporting methods have non-response rates in the range of 30-50 percent.

Following the approach in Hungnes et al. (2024), we analyze four distinct demographic groups, also referred to as domains: (i) men aged 15-24, (ii) men aged 25-74, (iii) women aged 15-24, and (iv) women aged 25-74. The distinction between younger and older workers is particularly relevant given that young individuals often experience lower employment rates and greater job instability. Additionally, although the gender gap in employment has narrowed over time, it remains appropriate to differentiate between men and women. In our analysis, we categorize individuals into three labour market categories: employed, unemployed, and outside the labour market.

Table 5: Estimates of the wave-specific effects

	Males, 15–24 y.o.		Males, 25–74 y.o.		Females, 15–24 y.o.		Females, 25-74 y.o.	
	unempl.	empl.	unempl.	empl.	unempl.	empl.	unempl.	empl.
$\lambda^1 * 100$	0.64	-1.60	0.31	-0.26	0.43	-2.30	0.37	-0.54
$\lambda^2 * 100$	0.05	-0.46	0.08	0.08	0.27	0.27	0.10	-0.04
$\lambda^3 * 100$	0.29	-0.22	-0.01	-0.08	-0.18	0.82	-0.06	0.02
$\lambda^4 * 100$	0.08	-0.11	-0.04	0.06	-0.15	0.27	-0.08	0.09
$\lambda^5 * 100$	0.13	0.37	-0.13	0.12	0.04	0.31	-0.12	0.13
$\lambda^6 * 100$	-0.35	0.40	-0.18	0.01	-0.43	0.45	-0.13	0.02
$\lambda^7 * 100$	-0.37	1.00	-0.08	-0.22	-0.06	0.26	-0.12	-0.10
$\lambda^8 * 100$	-0.47	0.62	0.04	0.29	0.08	-0.09	0.03	0.42

Notes: Estimates of the wave-specific effects according to (23).

8.1 Estimates of the parameters needed for the composite estimator

Table 4 presents descriptive statistics for the labour market categories in the domains. These statistics reveal significant differences in employment rates between age groups. For men, the employment rate is approximately 51 percent for those aged 15-24, compared to about 75 percent for those aged 25-74. The unemployment rate is lower for the older age group, and the participation in the labour force, defined as the sum of employed and unemployed individuals, is also higher among older men, at 78 percent, compared to 57 percent for younger men. Similar trends are observed for women, with the labour force participation rate for young women roughly matching that of young men. However, the composition of the labour force differs slightly between domains, with employed women in the older age group having an employment rate approximately 7 percentage points lower than men in the same age group.

In Table 5, we report the wave-specific effects under the assumption that these effects are time-invariant. For unemployment, a positive wave-specific effect is observed for wave 1 across all domains. It is well-established that a higher proportion of people are classified as unemployed when they first participate in the LFS compared to subsequent waves (see also Section 2). For men aged 25-74 years, this effect is the smallest. Specifically, the over-representation of unemployed individuals in wave 1 for this group is 0.31 percent of the domain population. Given that $2.32 + 75.50 = 77.82$ percent of this population participates in the labour force (see Table 4), this translates into an unemployment rate in wave 1 that is $0.31/0.78 \approx 0.4$ percentage points higher than the average across all waves.

For young men, the over-representation of unemployed individuals in wave 1 is 0.64 percent. With a labour force participation rate of $5.96 + 50.97 = 56.93$ percent for this domain, this implies that the unemployment rate, relative to the labour force, is $0.64/0.57 = 1.1$ percentage points higher in wave 1 compared to the average of all waves.

For employed individuals, we observe a clear under-representation in wave 1 across all domains. Additionally, there is a substantial over-representation of individuals outside the workforce in wave 1, though this is not directly reported in Table 5. It can be inferred residually, as the wave-specific effects across categories must sum to zero.

Table 6: Other estimates

	Males, 15–24 y.o.	Males, 25–74 y.o.	Females, 15–24 y.o.	Females, 25-74 y.o.
$\mathbf{\Phi}$	$\begin{pmatrix} 0.11 & -0.02 \\ 0.26 & 0.64 \end{pmatrix}$	$\begin{pmatrix} 0.20 & -0.06 \\ 0.26 & 0.79 \end{pmatrix}$	$\begin{pmatrix} 0.04 & -0.05 \\ 0.14 & 0.60 \end{pmatrix}$	$\begin{pmatrix} 0.26 & -0.03 \\ 0.28 & 0.81 \end{pmatrix}$
$\mathbf{\Omega} * 100^2$	$\begin{pmatrix} 7.94 & -3.60 \\ -3.60 & 35.86 \end{pmatrix}$	$\begin{pmatrix} 0.87 & -0.77 \\ -0.77 & 4.94 \end{pmatrix}$	$\begin{pmatrix} 7.19 & -4.26 \\ -4.26 & 34.63 \end{pmatrix}$	$\begin{pmatrix} 0.61 & -0.44 \\ -0.44 & 6.03 \end{pmatrix}$
$\boldsymbol{\alpha}$	$\begin{pmatrix} 0.09 & -0.02 \\ 0.24 & 0.59 \end{pmatrix}$	$\begin{pmatrix} 0.17 & -0.06 \\ 0.25 & 0.74 \end{pmatrix}$	$\begin{pmatrix} 0.04 & -0.05 \\ 0.13 & 0.54 \end{pmatrix}$	$\begin{pmatrix} 0.23 & -0.03 \\ 0.27 & 0.77 \end{pmatrix}$
$\boldsymbol{\beta}$	$\begin{pmatrix} 0.08 & -0.01 \\ 0.12 & 0.22 \end{pmatrix}$	$\begin{pmatrix} 0.15 & -0.01 \\ 0.07 & 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.04 & -0.02 \\ 0.08 & 0.23 \end{pmatrix}$	$\begin{pmatrix} 0.17 & -0.00 \\ 0.12 & 0.16 \end{pmatrix}$

Notes: Estimation results when the first element of μ is unemployed people and the second element is employed people, both measured as a share of the total population. Estimates according to (24), (25), (17), and (18).

In Table 6, we report the estimates of the parameters needed for the multivariate composite estimator. We note that for the estimate of $\boldsymbol{\alpha}$ as given in (17), the term $\mathbf{\Phi\Omega\Phi}^\top\mathbf{\Omega}^{-1}$ is almost negligible such that the estimate of $\boldsymbol{\alpha}$ is essentially a down-scaled version of the estimate of $\mathbf{\Phi}$: specifically, $\widetilde{\boldsymbol{\alpha}} \approx \frac{J-1}{J}\widetilde{\mathbf{\Phi}}$. Similarly, the effect of how wave 1 deviates from the direct estimate of $\boldsymbol{\mu}_t$, i.e. the last term in (15), is almost negligible, as this effect is scaled down by the number of waves (which is 8 for Norway), meaning that the impact of wave 1 on the overall average is given by the estimate of $\boldsymbol{\beta} \cdot \frac{1}{8}$.

The first element of $\boldsymbol{\mu}_t$ corresponds to the number of unemployed persons, expressed as a share of the total population. The estimates for unemployment in the first row of $\boldsymbol{\alpha}$ are most relevant here. For both young men and women, they are close to zero: for young females, $\tilde{\alpha}_{1,1} = 0.04$ and $\tilde{\alpha}_{1,2} = -0.05$; for young men, $\tilde{\alpha}_{1,1} = 0.09$ and $\tilde{\alpha}_{1,2} = -0.02$. Given these small estimates, the composite estimator places almost full weight on the direct estimate. Hence, the direct estimate provided by (16) will be a good estimate of the share of unemployed for young individuals.

For older individuals of both sexes, the estimate of $\alpha_{1,2}$ is close to zero, while the estimate of $\alpha_{1,1}$ is positive. This implies that the multivariate estimator in (15) will produce estimates of unemployment that are close to those of a univariate composite estimator (as in (9)). Therefore, the benefits of using the multivariate composite estimator for unemployment may be limited in this case.

The second element of $\boldsymbol{\mu}_t$ corresponds to the number of employed persons, also expressed as a share of the total population. For the estimates of employed persons, the estimates in the second row of $\boldsymbol{\alpha}$ are the relevant ones. These estimates are substantial: for all domains, $\tilde{\alpha}_{2,2} > 0.5$, and for most domains, $\tilde{\alpha}_{2,1} \approx 0.25$ (except for young females, where the estimate is about half of that in other domains). These findings underscore two key insights: first, the persistence of survey errors, particularly in employment estimates; and second, the necessity of considering interrelationships between employment, unemployment, and labour market non-participation in the composite estimator.

To interpret the implications of these estimates for the multivariate composite estimator, we apply the version of the multivariate composite estimator in (20), which in a system with the groups unemployment and employment as modelled becomes

$$\begin{pmatrix} \hat{\mu}_{u,t} - \hat{\mu}^{dir}_{u,t} \\ \hat{\mu}_{e,t} - \hat{\mu}^{dir}_{e,t} \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}_{uu} & \tilde{\alpha}_{ue} \\ \tilde{\alpha}_{eu} & \tilde{\alpha}_{ee} \end{pmatrix} \begin{pmatrix} \hat{\mu}_{u,t-l} - \hat{\mu}^{dir}_{u,t-l} \\ \hat{\mu}_{e,t-l} - \hat{\mu}^{dir}_{e,t-l} \end{pmatrix} + \begin{pmatrix} \tilde{\xi}_{u,t} \\ \tilde{\xi}_{e,t} \end{pmatrix},$$

where ´u' and ´e' indicate unemployment and employment, respectively. Later, we will use ´o' to indicate for those outside the labour force. Furthermore, for simplicity, we assume that $\boldsymbol{y}(\lambda)^{8}_{t-l} \approx \widehat{\boldsymbol{\mu}}(\lambda)^{dir}_{t-l}$ such that we can treat $\boldsymbol{\xi}_t$ as innovations with expectation equal to zero.

Within this framework, we consider the case in which the composite estimate for employment is lower than the direct estimate at the last time they participated in the survey, i.e. $\hat{\mu}_{e,t-l} - \hat{\mu}^{dir}_{e,t-l} < 0$. The effect of this on $\hat{\mu}_{e,t} - \hat{\mu}^{dir}_{e,t}$ will depend on whether the deviation between the composite and direct estimate of employment is offset by a corresponding deviation in either unemployment or for those outside the labour force:

- Consider first that the composite estimate for unemployment is equal to the direct estimate in the previous period the participants are interviewed, i.e. $\hat{\mu}_{u,t-l} - \hat{\mu}^{dir}_{u,t-l} = 0$. From this it follows that the composite estimate for those outside the labour force must exceed the direct estimate in that period, such that $\left(\hat{\mu}_{o,t-l} - \hat{\mu}^{dir}_{o,t-l}\right) = -\left(\hat{\mu}_{e,t-l} - \hat{\mu}^{dir}_{e,t-l}\right) > 0$. The implication for the composite estimate of employment in the current period is $E\left[\hat{\mu}_{e,t} - \hat{\mu}^{dir}_{e,t} | \hat{\mu}_{u,t-l} - \hat{\mu}^{dir}_{u,t-l} = 0\right] = \tilde{\alpha}_{ee}\left(\hat{\mu}_{e,t-l} - \hat{\mu}^{dir}_{e,t-l}\right)$. As the estimate of α_{ee} is typically large (from 0.54 to 0.77), this implies a high persistence in the deviation between the composite estimate and the direct estimate of employment.

- Consider second that it is the composite estimate for those outside the labour force that is equal to its direct estimate in period $t-l$, i.e. $\hat{\mu}_{o,t-l} - \hat{\mu}^{dir}_{o,t-l} = 0$. Then the deviation between the estimates for employed in period $t-l$ must be offset by a deviation between the estimates for unemployed in the same period, i.e. $\left(\hat{\mu}_{u,t-l} - \hat{\mu}^{dir}_{u,t-l}\right) = -\left(\hat{\mu}_{e,t-l} - \hat{\mu}^{dir}_{e,t-l}\right)$. The implication for the composite estimate for employment in the current period is now that $E\left[\hat{\mu}_{e,t} - \hat{\mu}^{dir}_{e,t} | \left(\hat{\mu}_{u,t-l} - \hat{\mu}^{dir}_{u,t-l}\right) = -\left(\hat{\mu}_{e,t-l} - \hat{\mu}^{dir}_{e,t-l}\right)\right] = (\tilde{\alpha}_{ee} - \tilde{\alpha}_{eu})\left(\hat{\mu}_{e,t-l} - \hat{\mu}^{dir}_{e,t-l}\right)$. Since the estimate of α_{eu} is positive (but smaller than α_{ee}), this implies a lower persistence in the deviation between the composite estimate and the direct estimate for employment.

The illustration above highlights the value of using a multivariate composite estimator, which distinguishes between various labour statuses. The multivariate estimator is particularly valuable for employment figures, as demonstrated by our empirical example, which underscores the importance of differentiating between unemployed individuals and individuals outside the labour force.

Table 7: Fraction of absolute deviations between the estimates of the multivariate composite estimator and the direct estimator ($\mu - \mu^{dir}$) that exceeds a particular percentage point

	Males, 15–24 y.o.			Males, 25–74 y.o.			Females, 15–24 y.o.			Females, 25-74 y.o.		
	u	e	o	u	e	o	u	e	o	u	e	o
$>$ 2 pp.	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$>$ 1 pp.	0.00	0.15	0.14	0.00	0.01	0.00	0.00	0.09	0.07	0.00	0.03	0.01
$>$ 0.5 pp.	0.00	0.53	0.50	0.00	0.15	0.10	0.00	0.39	0.36	0.00	0.36	0.34
$>$ 0.4 pp.	0.00	0.67	0.64	0.00	0.24	0.20	0.00	0.51	0.47	0.00	0.47	0.45
$>$ 0.3 pp.	0.00	0.74	0.74	0.00	0.40	0.32	0.00	0.63	0.58	0.00	0.61	0.60
$>$ 0.2 pp.	0.00	0.80	0.81	0.00	0.62	0.55	0.01	0.75	0.75	0.00	0.74	0.72
$>$ 0.1 pp.	0.02	0.91	0.88	0.04	0.81	0.79	0.10	0.86	0.85	0.01	0.88	0.87
$>$ 0.05 pp.	0.25	0.95	0.97	0.29	0.92	0.88	0.37	0.92	0.91	0.16	0.94	0.95
$>$ 0.04 pp.	0.38	0.97	0.97	0.40	0.93	0.90	0.48	0.93	0.92	0.23	0.95	0.95
$>$ 0.03 pp.	0.51	0.98	0.98	0.54	0.94	0.93	0.58	0.93	0.92	0.39	0.97	0.97
$>$ 0.02 pp.	0.69	0.99	0.98	0.68	0.94	0.95	0.73	0.96	0.96	0.56	0.97	0.98
$>$ 0.01 pp.	0.81	0.99	0.99	0.82	0.97	0.96	0.90	0.98	0.97	0.78	0.98	1.00

Notes: A row in the table shows the proportion of monthly LFS figures for the period April 2006 – December 2020 (as the figures with the multivariate composite estimator equals the direct estimator the three first months of the sample) that are revised more than indicated in the left column. 'u' indicates 'unemployment'; 'e' indicates 'employment'; and 'o' indicates 'outside labour force'.

8.2 Effect of using the composite estimator

We can assess the impact of using the multivariate composite estimator instead of the direct estimator using (20). Table 7 compares the revisions that would result in the LFS figures for the unemployed, employed, and those outside the labour market across different domains using the multivariate composite estimator instead of the direct estimator.

The differences in LFS figures for the unemployed, calculated with the multivariate composite estimator instead of the direct estimator, are minimal. For historical data, only one monthly observation (representing 0.01 or 1 percent in Table 7) shows a revision exceeding 0.2 percentage points, specifically for young females. Only a few months see a change in the estimated unemployment rate greater than 0.1 percentage points with the multivariate composite estimator: 2 percent for young men; 4 percent for older men; 10 percent for young women; and 1 percent for women in the oldest age group.

The transition from a direct estimator to a multivariate composite estimator will result in notable revisions for the employment figures. For example, 15 percent of monthly figures for young men will be revised by more than 1 percentage point, highlighting the estimator's capacity to refine estimates for this demographic group. For older men, only one monthly figure will be revised by at least 1 percentage point. For women, 9 percent of the monthly figures for young women and 3 percent for older women are revised by one percentage point or more. Most of the monthly employment figures see revisions of at least 0.1 percentage points: 91 percent for young men; 81 percent for older men; 86 percent for young women; and 88 percent for older women.

Revisions for those outside the labour force are somewhat smaller than for employment figures. This is because revisions in unemployment figures partly offset changes in employment figures,

Table 8: Estimation results regarding time-varying wave-specific effects.

	Males, 15–24 y.o.	Males, 25–74 y.o.	Females, 15–24 y.o.	Females, 25-74 y.o.
$\sigma_V^2 * 10^2$	0.38	0.05	0.035	0.06
$\sigma_L^2 * 10^{12}$	0.52	0.11	0.06	0.08
$q * 10^{10}$	1.38	2.24	0.17	1.24
$\bar{p} * 10^5$	1.17	1.50	0.41	1.11
$\bar{k} * 10^2$	1.17	1.50	0.41	1.11

Notes: The table reports estimates of σ_I^2, see (29); σ_L^2, see (30); $q = \sigma_L^2/\sigma_V^2$; $\bar{p}$, see (32); and $\bar{k}$, see (33).

resulting in generally somewhat smaller impacts on figures for those outside the labour market.

8.3 Time-varying wave-specific effects?

When constructing the $(I-1)(J-1) = 14$ independent processes for the wave-specific effects and estimating the hyperparameters σ_V^2 and σ_L^2 in (29) and (30), we find that the signal-to-noise ratio ($q = \sigma_L^2/\sigma_V^2$) is below 10^{-9} for all domains; see Table 8. This corresponds to an estimate of $\bar{k}$ less than 10^{-4}, strongly suggesting that the wave-specific effects are effectively time-invariant.

Consequently, if estimates of wave-specific effects are updated with new observations, the appropriate weighting factor should be $1/(180+1)$. This represents the weight assigned to an individual observation when estimating a time-invariant wave-specific effect.

9 Conclusions

This paper has introduced a multivariate composite estimator for the Labour Force Survey (LFS), which accounts for autocorrelation in the sampling errors across various labour market categories. By addressing this autocorrelation, the estimator enhances the accuracy of population estimates. While the population estimates produced by the proposed estimator are not seasonally adjusted, they are free from autocorrelated sampling errors. This property ensures that these estimates can be reliably used to generate both seasonally adjusted time series and trend estimates without the risk of introducing spurious trends.

The multivariate composite estimator proposed in this paper shares several similarities with the estimator introduced by Merkouris (2024). While the estimator in Merkouris (2024) incorporates the weighting of survey respondents, it does not account for wave-specific biases. In contrast, the estimator presented here explicitly incorporates wave-specific biases and allows these effects to evolve over time. This key distinction enhances the flexibility and accuracy of our approach. By modelling wave-specific effects, our estimator also helps mitigate potential distortions caused by changes in survey design, such as those resulting from the restructuring of the LFS in 2020/2021.

An important practical contribution of this study is the demonstration of the estimator's impact on real-world LFS data from Norway. Our empirical results illustrate that revisions to unemployment estimates are relatively small when transitioning from the direct estimator to the multivariate composite estimator. However, significant revisions are observed for employment figures, particularly for younger demographic groups, reflecting the benefits of explicitly modelling interdependencies among labour market statuses. These findings emphasize the ability of the multivariate composite estimator to improve the reliability of employment statistics, especially for smaller subgroups that are more prone to sampling variability.

References

Bailar, B. A. (1975). The effects of rotation group bias on estimates from panel surveys. *Journal of the American Statistical Association*, 70(349):23–30.

Breau, P. and Ernst, L. (1983). Alternative Estimators to the Current Composite Estimator. *Proceedings of the Section on Survey Research Methods, American Statistical Association*, 397402:397–402.

Elliott, D. J. and Zong, P. (2019). Improving timeliness and accuracy of estimates from the UK labour force survey. *Statistical Theory and Related Fields*, 3(2):186–198.

Eurostat (2014). *Labour force surveys in the EU, candidate and EFTA countries: Main characteristics of national surveys, 2013.*

Eurostat (2022). *Quality report of the European Union Labour Force Survey 2020.*

Fuller, W. A. (1990). Analysis of repeated surveys. *Survey Methodology*, 16(2).

Fuller, W. A. and Rao, J. N. K. (2001). A regression composite estimator with application to the Canadian Labour Force Survey. *Survey Methodology*, 27(1):45–51.

Gonçalves, C., Hidalgo, L., Silva, D., and den Brakel, J. (2022). Single-Month Unemployment Rate Estimates for the Brazilian Labour Force Survey Using State-Space Models. *Journal of the Royal Statistical Society Series A: Statistics in Society*, 185(4):1707–1732.

Gurney, M. and Daly, J. F. (1965). A Multivariate Approach To Estimation In Periodic Sample Surveys. *American Statistical Association Proceedings of the Social Statistics Section 8*, pages 242–257.

Hansen, M. H., Hurwitz, W. N., Nisselson, H., and Steinberg, J. (1955). The Redesign of the Census Current Population Survey. *Journal of the American Statistical Association*, 50(271):701–719.

Harvey, A. and Chung, C.-H. (2000). Estimating the Underlying Change in Unemployment in the UK. *Journal of the Royal Statistical Society Series A: Statistics in Society*, 163(3):303–309.

Harvey, A. C. (1989). *Forecasting, Structural Time Series Models and the Kalman Filter.* Cambridge University Press.

Hausman, J. A. and Watson, M. W. (1985). Errors in variables and seasonal adjustment procedures. *Journal of the American Statistical Association*, 80(391):531–540.

Hungnes, H., Skjerpen, T., Hamre, J. I., Jansen, X. C., Pham, D. Q., and Sandvik, O. (2024). Structural Break in the Norwegian Labor Force Survey Due to a Redesign During a Pandemic. *Journal of Official Statistics*, 40(1):122–160.

Jessen, R. J. (1942). Statistical investigation of a sample survey for obtaining farm facts. *Research Bulletin (Iowa Agriculture and Home Economics Experiment Station)*, 26(304).

Krueger, A. B., Mas, A., and Niu, X. (2017). The Evolution of Rotation Group Bias: Will the Real Unemployment Rate Please Stand Up? *The Review of Economics and Statistics*, 99(2):258–264.

Mayer, A. (2018). Improving seasonal adjustment by accounting for sample error correlation using state space models. *Statistical Journal of the IAOS*, 34(3):409–423.

Merkouris, T. (2024). A New Approach to Composite Estimation for Repeated Surveys with Rotating Panels. *Journal of Official Statistics*, 40(3):409–424.

Patterson, H. D. (1950). Sampling on Successive Occasions with Partial Replacement of Units. *Journal of the Royal Statistical Society: Series B (Methodological)*, 12(2):241–255.

Pfeffermann, D. (1991). Estimation and seasonal adjustment of population means using data from repeated surveys. *Journal of Business and Economic Statistics*, 9(2):163–175.

Pfeffermann, D., Feder, M., and Signorelli, D. (1998). Estimation of autocorrelations of survey errors with application to trend estimation in small areas. *Journal of Business and Economic Statistics*, 16(3):339–348.

Schiavoni, C., Koopman, S. J., Palm, F., Smeekes, S., and van den Brakel, J. (2024). Time-varying correlations in multivariate unobserved components time series models. *Journal of the Royal Statistical Society Series A: Statistics in Society*.

Schiavoni, C., Palm, F., Smeekes, S., and van den Brakel, J. (2021). A dynamic factor model approach to incorporate Big Data in state space models for official statistics. *Journal of the Royal Statistical Society. Series A: Statistics in Society*, 184(1):324–353.

Scott, A. J. and Smith, T. M. F. (1974). Analysis of Repeated Surveys Using Time Series Methods. *Journal of the American Statistical Association*, 69(347):674–678.

Scott, A. J., Smith, T. M. F., and Jones, R. G. (1977). The Application of Time Series Methods to the Analysis of Repeated Surveys. *International Statistical Review*, 45(1):13–28.

Silva, D. B. N. and Smith, T. M. F. (2001). Modelling Compositional Time Series from Repeated Surveys. *Survey Methodology*, 27(12001):205–215.

Singh, A. C., Kennedy, B., and Wu, S. (2001). Regression Composite Estimation for the Canadian Labour Force Survey with a Rotating Panel Design. *Survey Methodology*, 27(1):33–44.

Singh, A. C., Kennedy, B., Wu, S., and Brisebois, F. (1997). Composite estimation for the Canadian Labour Force Survey. *Proceedings of the Survey Research Methods Section*, pages 300–305.

Singh, A. C. and Merkouris, P. (1995). Composite estimation by modified regression for repeated surveys. *Proceedings of the Survey Research Methods Section: American Statistical Association*, pages 420–425.

Statistics Bureau of Japan (2023). Sampling Method, Estimation Method, and Sampling Errors of Estimates.

Statistics Canada (2017). *Methodology of the Canadian Labour Force Survey*. Number 71.

Stats NZ Tatauranga Aotearoa (2023). About the Household Labour Force Survey.

Steel, D. and McLaren, C. (2009). Design and Analysis of Surveys Repeated over Time. In *Handbook of Statistics*, volume 29, pages 289–313.

Stephan, F. F., Frankel, L. R., and Teper, L. (1954). The measurement of employment and unemployment by the Bureau of the census in its Current Population Survey. *Report of the Special Advisory Committee on Employment Statistics*.

U.S. Census Bureau (2019). Design and Methodology: Current Population Survey–America's Source for Labor Force Data. *U.S. Census Bureau*, (Technical Paper 77):1–175.

van den Brakel, J. and Krieg, S. (2009). Estimation of the monthly unemployment rate through structural time series modelling in a rotating panel design. *Survey Methodology*, 35(2):177–190.

van den Brakel, J. and Krieg, S. (2016). Small Area Estimation with State Space Common Factor Models for Rotating Panels. *Journal of the Royal Statistical Society Series A: Statistics in Society*, 179(3):763–791.

van den Brakel, J. and Roels, J. (2010). Intervention analysis with state-space models to estimate discontinuities due to a survey redesign. *Annals of Applied Statistics*, 4(2):1105–1138.

van den Brakel, J., Souren, M., and Krieg, S. (2022). Estimating Monthly Labour Force Figures During the COVID-19 Pandemic in the Netherlands. *Journal of the Royal Statistical Society Series A: Statistics in Society*, 185(4):1560–1583.

Whittle, P. (1963). On the Fitting of Multivariate Autoregressions, and the Approximate Canonical Factorization of a Spectral Density Matrix. *Biometrika*, 50(1):129–134.

Yule, G. U. (1907). On the Theory of Correlation for any Number of Variables, treated by a New System of Notation. *Proceedings of the Royal Society of London, Series A*, 79:182–193.

Zhang, X. M., Brakel, J. A. v. d., and Tam, S.-M. (2019). Using state space models as a statistical impact measurement of survey redesigns: a case study of the labour force survey of the Australian Bureau of Statistics. *Statistical Theory and Related Fields*, 3(2):224–238.

A Appendix

The appendix provides a detailed derivation of the multivariate composite estimator in (15). The system in (13) and (14) for $t > l$ can be written as:

$$\begin{aligned} \boldsymbol{y}_t^1(\lambda) &= \boldsymbol{\mu}_t + \boldsymbol{\varepsilon}_t^1 && \text{for } j = 1 \\ \boldsymbol{y}_t^j(\lambda) &= \boldsymbol{\mu}_t + \boldsymbol{\Phi}\left(\boldsymbol{y}_{t-l}^{j-1}(\lambda) - \boldsymbol{\mu}_{t-l}\right) + \boldsymbol{K}\boldsymbol{\varepsilon}_t^j && \text{for } j > 1 \end{aligned} \tag{A.1}$$

The first order condition can be found by premultiplying this system with the estimates of the covariance matrix given by $\boldsymbol{\Omega}^{-1}$ for $j = 1$ and $\left(\boldsymbol{K}\boldsymbol{\Omega}\boldsymbol{K}^\top\right)^{-1}$ for $j = 2, \ldots, J$, and then taking the sum of the expectations (over j):

$$\begin{aligned} &\widetilde{\boldsymbol{\Omega}}^{-1}\boldsymbol{y}(\lambda)_t^1 + \left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\sum_{j=2}^{J}\boldsymbol{y}(\lambda)_t^j \\ =&\left[\widetilde{\boldsymbol{\Omega}}^{-1} + (J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\right]\widehat{\boldsymbol{\mu}}_t + \left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\widetilde{\boldsymbol{\Phi}}\sum_{j=2}^{J}\left(\boldsymbol{y}(\lambda)_{t-l}^{j-1} - \widehat{\boldsymbol{\mu}}_{t-l}\right). \end{aligned}$$

Moving the term involving $\widehat{\boldsymbol{\mu}}_t$ to the left-hand side yields:

$$\begin{aligned} &\left[\widetilde{\boldsymbol{\Omega}}^{-1} + (J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\right]\widehat{\boldsymbol{\mu}}_t \\ =&\widetilde{\boldsymbol{\Omega}}^{-1}\boldsymbol{y}(\lambda)_t^1 + \left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\sum_{j=2}^{J}\boldsymbol{y}(\lambda)_t^j \\ &+ (J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\widetilde{\boldsymbol{\Phi}}\left(\widehat{\boldsymbol{\mu}}_{t-l} - \frac{1}{J-1}\sum_{j=2}^{J}\boldsymbol{y}(\lambda)_{t-l}^{j-1}\right) \\ =&\widetilde{\boldsymbol{\Omega}}^{-1}\boldsymbol{y}(\lambda)_t^1 + \left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\right)\sum_{j=2}^{J}\boldsymbol{y}(\lambda)_t^j \\ &+ (J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\widetilde{\boldsymbol{\Phi}}\left(\widehat{\boldsymbol{\mu}}_{t-l} + \frac{1}{J-1}\sum_{j=2}^{J}\left(\boldsymbol{y}(\lambda)_t^j - \boldsymbol{y}(\lambda)_{t-l}^{j-1}\right)\right). \end{aligned} \tag{A.2}$$

By using $\boldsymbol{y}(\lambda)_t^j = \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} + \boldsymbol{y}(\lambda)_t^j - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}$ in the first term in the last equality, we have

$$\begin{aligned}
&\widetilde{\boldsymbol{\Omega}}^{-1}\boldsymbol{y}(\lambda)_t^1 + \left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\right)\sum_{j=2}^{J}\boldsymbol{y}(\lambda)_t^j \\
=&\widetilde{\boldsymbol{\Omega}}^{-1}\left(\widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} + \boldsymbol{y}(\lambda)_t^1 - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right) \\
&+ \left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\right)\sum_{j=2}^{J}\left(\widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} + \boldsymbol{y}(\lambda)_t^j - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right) \\
=&\left[\widetilde{\boldsymbol{\Omega}}^{-1} + (J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\right)\right]\widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} \\
&+ \widetilde{\boldsymbol{\Omega}}^{-1}\left(\boldsymbol{y}(\lambda)_t^1 - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right) + \left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\right)\sum_{j=2}^{J}\left(\boldsymbol{y}(\lambda)_t^j - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right) \\
=&\left[\widetilde{\boldsymbol{\Omega}}^{-1} + (J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\right)\right]\widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} \\
&+ \left[\widetilde{\boldsymbol{\Omega}}^{-1} - \left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\right)\right]\left(\boldsymbol{y}(\lambda)_t^1 - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right),
\end{aligned}$$

where $\sum_{j=2}^{J}\left(\boldsymbol{y}(\lambda)_t^j - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right) = -\left(\boldsymbol{y}(\lambda)_t^1 - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right)$ is used in the last equality.

By using this in (A.2), we can work out the coefficient matrix of the direct estimator:

$$\begin{aligned}
&\left[\widetilde{\boldsymbol{\Omega}}^{-1} + (J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\right]\widehat{\boldsymbol{\mu}}_t \\
=&\left[\widetilde{\boldsymbol{\Omega}}^{-1} + (J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\right)\right]\widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} \\
&+ (J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\widetilde{\boldsymbol{\Phi}}\left(\widehat{\boldsymbol{\mu}}_{t-l} + \frac{1}{J-1}\sum_{j=2}^{J}\left(\boldsymbol{y}(\lambda)_t^j - \boldsymbol{y}(\lambda)_{t-l}^{j-1}\right)\right) \\
&+ \frac{1}{J-1}\left[(J-1)\left(\widetilde{\boldsymbol{\Omega}}^{-1} - \left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\right)\right)\right]\left(\boldsymbol{y}(\lambda)_t^1 - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right), \qquad \text{(A.3)}
\end{aligned}$$

or

$$\widehat{\boldsymbol{\mu}}_t = \left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\alpha}}\right)\widehat{\boldsymbol{\mu}}(\lambda)_t^{dir} + \widetilde{\boldsymbol{\alpha}}\left[\widehat{\boldsymbol{\mu}}_{t-l} + \frac{1}{J-1}\sum_{j=2}^{J}\left(\boldsymbol{y}(\lambda)_t^j - \boldsymbol{y}(\lambda)_{t-l}^{j-1}\right)\right] + \widetilde{\boldsymbol{\beta}}\frac{1}{J-1}\left[\boldsymbol{y}(\lambda)_t^1 - \widehat{\boldsymbol{\mu}}(\lambda)_t^{dir}\right],$$

where, by using $\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1} = \left(\widetilde{\boldsymbol{\Omega}} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}'\right)^{-1}$, we have

$$\begin{aligned}
\widetilde{\boldsymbol{\alpha}} &= \left[\widetilde{\boldsymbol{\Omega}}^{-1} + (J-1)\left(\widetilde{\boldsymbol{\Omega}} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\right)^{-1}\right]^{-1}\left[(J-1)\left(\widetilde{\boldsymbol{\Omega}} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\right)^{-1}\widetilde{\boldsymbol{\Phi}}\right] \\
&= \left[J\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\right]^{-1}\left[(J-1)\widetilde{\boldsymbol{\Phi}}\right] \qquad \text{(A.4)} \\
\widetilde{\boldsymbol{\beta}} &= \left[\widetilde{\boldsymbol{\Omega}}^{-1} + (J-1)\left(\widetilde{\boldsymbol{\Omega}} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\right)^{-1}\right]^{-1}\left[(J-1)\left[\widetilde{\boldsymbol{\Omega}}^{-1} - \left(\widetilde{\boldsymbol{\Omega}} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\right)^{-1}\left(I - \widetilde{\boldsymbol{\Phi}}\right)\right]\right] \\
&= \left[J\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\right]^{-1}\left[(J-1)\widetilde{\boldsymbol{\Phi}}\left(\boldsymbol{I}_{I-1} - \widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\right)\right]. \qquad \text{(A.5)}
\end{aligned}$$

The second equalities in (A.4) and (A.5) can be shown by pre-multiplying (A.3) with $\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top$ and applying the identity $\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top = \widetilde{\boldsymbol{\Omega}} - \widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top$, as detailed below. For the (inverted) expression in the first square brackets for the estimates of both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (i.e., (A.4) and (A.5)), we have

$$\begin{aligned}\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)\left[\widetilde{\boldsymbol{\Omega}}^{-1}+(J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\right] &= \widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}+(J-1)\boldsymbol{I}_{I-1}\\ &= \left(\widetilde{\boldsymbol{\Omega}}-\widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\right)\widetilde{\boldsymbol{\Omega}}^{-1}+(J-1)\boldsymbol{I}_{I-1}\\ &= \boldsymbol{I}_{I-1}-\widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}+(J-1)\boldsymbol{I}_{I-1}\\ &= J\boldsymbol{I}_{I-1}-\widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}.\end{aligned}$$

For the expression in the second square brackets for the estimate of $\boldsymbol{\alpha}$, we have

$$\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)(J-1)\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\widetilde{\boldsymbol{\Phi}}=(J-1)\widetilde{\boldsymbol{\Phi}}.$$

For the expression in the second square brackets for the estimate of $\boldsymbol{\beta}$ when ignoring the scaling factor $(J-1)$, we have

$$\begin{aligned}\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)\left[\left(\widetilde{\boldsymbol{\Omega}}^{-1}-\left(\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\right)^{-1}\left(\boldsymbol{I}_{I-1}-\widetilde{\boldsymbol{\Phi}}\right)\right)\right] &= \left[\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{K}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}-\left(\boldsymbol{I}_{I-1}-\widetilde{\boldsymbol{\Phi}}\right)\right]\\ &= \boldsymbol{I}_{I-1}-\widetilde{\boldsymbol{\Phi}}\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}-\boldsymbol{I}_{I-1}+\widetilde{\boldsymbol{\Phi}}\\ &= \widetilde{\boldsymbol{\Phi}}\left[\boldsymbol{I}_{I-1}-\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{\Phi}}^\top\widetilde{\boldsymbol{\Omega}}^{-1}\right].\end{aligned}$$