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Small continuous surveys and the Kalman filter

Abstract:
The time series nature of repeated surveys is seldom taken into account. I present a statistical model of repeated surveys and construct a computationally feasible estimator based on the Kalman filter. The novelty is that the estimator efficiently uses the whole underlying data set. However, for computational purposes, we only need the first and second empirical moments of the data.

Keywords: Surveys, Kalman filter, time series.

JEL classification: C22, C53, C81.

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1 Introduction

A number of statistical series are estimated on the basis of surveys that are repeated regularly. The most common approach is to publish parameter estimates at regular intervals, say each year, pooling surveys collected throughout the year but ignoring previous years. However, it is natural to assume that most parameters of interest evolve slowly and smoothly. As this approach ignores within period variation and previous observations, it is an inefficient use of the data.

The use of times series techniques to improve results from repeated surveys was suggested by Jessen (1942) and studied in more detail by Gurney and Daly (1965). The methodology was further improved by Blight and Scott (1973) and Scott and Smith (1974) who suggest using statistical signal extraction methods to filter the time specific estimates of the parameters of interest. See e.g. the survey by Binder and Hidiroglou (1988) for further details on subsequent developments within this tradition. A more general theory of signal extraction using the Kalman filter was suggested by Tam (1987) and further developed by e.g. Binder and Dick (1989), Harvey and Chung (2000), and Pfeffermann (1991). The most common approach is to estimate a parameter such as the mean on each individual survey and then apply the Kalman filter on the estimates. However, there is a potential important loss of efficiency as a lot of information contained in each cross section may be lost by this two step procedure. A more satisfactory approach, which is the one taken by Tam (1987), is to integrate the time series model and the modelling of the individual observations.

However, if we use the ordinary Kalman filter algorithm, this will lead to extremely large matrices that has to be inverted hence causing severe computational problems unless each survey is extremely small. In the present work I use an approach relatively similar to Tam's and show how the Kalman filter algorithm may be transformed to fit estimation on repeated surveys without running into computational problems. It turns out that to estimate the mean of the population, we only need the empirical first and second moments in each period, so both the computational burden and the data requirements are small.

The model is presented in Section 2 and the computationally feasible version of the Kalman filter suitable for the model in Section 3. The likelihood function of the problem and different strategies for estimation of the parameters of the model are discussed in Section 4. Section 5 concludes. Some lengthy proofs are left to Appendix A whereas Appendix B outlines a computer program to implement the routine.

2 Model framework

We study a series of repeated surveys where it is assumed that the parameters of interest change relatively smoothly over time. We will present a model that makes this process more explicit. However, instead of modelling the process of the period averages, we shall rather model the evolution of each individual observation. This will assure efficient use of the data.

At a survey date $t \in (1, \ldots, T)$ we observe N_t individuals. I assume that observations are independent both within and between surveys. It is probably possible to extend the approach to repeated observations of each individual, but that is outside the scope of the present paper. Let y_{it} denote the m-vector of observations on individual i at time t. We are going to focus on estimating averages of the y_{it}'s. We may write

$$y_{it} = \mu_{it} + \varepsilon_{it} \tag{1}$$

where $\varepsilon_{ij} \sim N(\mathbf{0}_{m\times 1}, \Sigma_t)$ denotes a stochastic vector of individual characteristics and possible sampling errors and $\mathbf{0}_{m\times 1}$ is a $m \times 1$ vector of zeros. The variable of interest is then μ_{it}. It is normally not particularly interesting do estimate a separate μ for every individual. One approach is to assume that the μ_{it}'s are the same for all the individuals at a particular date, but there are also cases where it is fruitful to group individuals into e.g. geographical regions or household types, and assume that every group has their own μ. This is the approach we will pursue herein. Assume that there are G such groups, and an associated μ_{gt} for all $g \in (1, \ldots, G)$ at every date.[1] It will be useful to consider the stacked vector of all the means at date t

$$\mu_t = \left(\mu'_{1t}, \ldots, \mu'_{Gt}\right)'. \tag{2}$$

Expression (1) may now be written as

$$y_{it} = J_{g(i)t}\mu_t + \varepsilon_{it} \tag{3}$$

where g is the function that associates to each individual i the group that it belongs to, and the $Gm \times m$ matrix

$$J_{gt} = \left(\mathbf{0}_{(g-1)m\times m} \;\vdots\; I_m \;\vdots\; \mathbf{0}_{(G-g)m\times m} \right) \tag{4}$$

selects the appropriate elements from the vector μ_t for individuals in group g. We make a slight abuse of notation by letting g denote both the function that associates to each individual i its group and a typical group.

It is probably reasonable to expect that μ_t does not make extreme changes over a relatively short period of time. Particularly, we shall assume that there is a n-vector α_t following a VAR(1) process with Gaussian white noise, i.e.

$$\alpha_t = F\alpha_{t-1} + \xi_t, \tag{5}$$

such that $\mu_t = Z\alpha_t$ where $\xi_t \sim N\left(\mathbf{0}_{n\times 1}, Q\right)$ and F is a $n \times n$ transition matrix. Since α_t is an unobserved vector, any finite-dimensional vector ARMA-process may be rewritten as such a VAR(1) process. Defining

$$\begin{aligned} J_t &= \left(J'_{g(1)t}, \ldots, J'_{g(N_t)t}\right)' && (6)\\ \tilde{\varepsilon}_t &= \left(\varepsilon'_{1t}, \ldots, \varepsilon'_{N_t t}\right)' && (7)\\ \tilde{y}_t &= \left(y'_{1t}, \ldots, y'_{N_t t}\right)', && (8) \end{aligned}$$

we can write the complete model as

$$\begin{aligned} \tilde{y}_t &= J_t Z\alpha_t + \tilde{\varepsilon}_t \\ \alpha_t &= F\alpha_{t-1} + \xi_t \\ \tilde{\varepsilon}_t &\sim N\left(\mathbf{0}_{N_t m\times 1}, I_{N_t} \otimes \Sigma_t\right) \\ \xi_t &\sim N\left(\mathbf{0}_n, Q\right) \\ \alpha_0 &\sim N\left(a_0, Q_0\right), \end{aligned} \tag{9}$$

where we also added assumptions about the distribution of the initial state α_0. Treating $J_t Z$ as a single matrix transforming the state vector into the expectation of the observed data, it is seen that this is a model on "almost standard" state space form[2].

3 The Kalman filter

Let us initially assume that we know the vector of hyper-parameters

$$\Theta = \left(\left\{\operatorname{vec}\left(\Sigma_t\right)'\right\}, \operatorname{vec}\left(Q\right)', a_0, \operatorname{vec}\left(Q_0\right)\right),$$

[1] The covariance matrix Σ_t is assumed the be identical for every group, but this assumption may easily be relaxed.

[2] The term almost standard is used since the dimension of $\tilde{y}_t$ varies with time. Nevertheless, replacing $\tilde{y}_t$ with $\dot{y}_t \equiv \left(\tilde{y}'_t \vdots \mathbf{0}_{1\times(\max_t N_t)-Nt}\right)'$ and J_t with $\dot{J}_t \equiv \left(J'_t \vdots \mathbf{0}_{1\times m[(\max_t N_t)-Nt]}\right)'$ would transform the model to standard state space form. It is easily seen that this will not change any of the results below.

as well as the transition matrix F and the matrices Z and J_t. An optimal estimate of the α's and the μ's may then be calculated by the means of the Kalman filter (see e.g. Fahrmeir and Tutz (1994, ch. 8), Hamilton (1995 ch. 13) or Harvey (1989) for overviews to the Kalman filter). At date t, the information set is defined as $\mathcal{Y}_t = (\tilde{y}_1', \ldots, \tilde{y}_t')'$. Let us denote the expectation of the vector α_{t_1} given the information set at date t_2 as

$$a_{t_1|t_2} \equiv E\left(\alpha_{t_1} \left| \mathcal{Y}_{t_2}\right.\right),$$

and its covariance matrix by

$$V_{t_1|t_2} = E\left[\left(\alpha_{t_1} - a_{t_1|t_2}\right)\left(\alpha_{t_1} - a_{t_1|t_2}\right)' \left|\mathcal{Y}_{t_2}\right.\right].$$

The Kalman filter is calculated by the following recursion:

$$\begin{aligned} a_{t|t-1} &= F a_{t-1|t-1} \\ V_{t|t-1} &= F V_{t-1|t-1} F' + Q \\ a_{t|t} &= a_{t|t-1} + K_t\left(\tilde{y}_t - J_t Z a_{t|t-1}\right) \\ V_{t|t} &= V_{t|t-1} - K_t J_t Z V_{t|t-1} \\ K_t &= V_{t|t-1} Z' J_t' \left(J_t Z V_{t|t-1} Z' J_t' + I_{N_t} \otimes \Sigma_t\right)^{-1}. \end{aligned} \tag{10}$$

In their current form, these formulae are not particularly useful for larger surveys since the vector $\tilde{y}_t$, and consequently the matrix $\left(J_t Z V_{t|t-1} Z' J_t' + I_{N_t} \otimes \Sigma_t\right)$, which is to be inverted, may be of very high dimension, and hence require large amounts of calculation. However, due to the data structure assumed above, it is shown in the appendix that the recursion in (10) may be written as

$$\begin{aligned} a_{t|t-1} &= F a_{t-1|t-1} \\ V_{t|t-1} &= F V_{t-1|t-1} F' + Q \\ V_{t|t} &= \left[V_{t|t-1}^{-1} + Z'\left(\mathcal{N}_t^G \otimes \Sigma_t^{-1}\right) Z\right]^{-1} \\ a_{t|t} &= a_{t|t-1} + V_{t|t} Z'\left(\mathcal{N}_t^G \otimes \Sigma_t^{-1}\right)\left(\bar{y}_t^G - Z a_{t|t-1}\right). \end{aligned} \tag{11}$$

In these expressions, $\bar{y}_t^G$ denotes the within group averages defined as

$$\bar{y}_t^G \equiv \begin{pmatrix} \frac{1}{N_1^g}\sum_{g(i)=1} y_{it} \\ \vdots \\ \frac{1}{N_G^g}\sum_{g(i)=G} y_{it} \end{pmatrix}. \tag{12}$$

The matrix $\mathcal{N}_t^G$ is the matrix with the number of members of each group at date t along the diagonal.

Using the recursion (11), we calculate estimates of α_t given the information set $\mathcal{Y}_t$. This is not normally optimal, since the complete information set $\mathcal{Y}_T$ normally contains more information about α_t than does $\mathcal{Y}_t$. To obtain estimates employing the full information set, we use the so-called Kalman smoother. Define the sequence of matrices

$$B_t = V_{t-1|t-1} F' V_{t|t-1}^{-1}. \tag{13}$$

The smoothed estimates of α are found by the backward recursion

$$a_{t-1|T} = a_{t-1|t-1} + B_t\left(a_{t|T} - a_{t|t-1}\right) \tag{14}$$

$$V_{t-1|T} = V_{t-1|t-1} + B_t\left(V_{t|T} - V_{t|t-1}\right) B_t'. \tag{15}$$

See e.g. Hamilton (1995: ch. 13) for a proof. Since all the expressions entering these expressions are of low dimensionality, no transformations are necessary for our purposes.

4 Estimation

The algorithm described above was based upon the knowledge of the hyper-parameters, as well as the matrices F and Z. Since most of these parameters are normally not known, they will have to be estimated. In the present work, I derive estimators for the hyper parameters, but assume that F and Z are known matrices. It is straightforward to extend the framework to allow for estimating selected parameters in these matrices.

In the present work, I will discuss estimation by the method of maximum likelihood (ML). This is the usual approach in Kalman filter models. The likelihood of the data given a set of parameter values is

$$f\left(\mathcal{Y}_T;\Theta\right)=f\left(\tilde{y}_1\right)f\left(\tilde{y}_2|\mathcal{Y}_1\right)\cdots f\left(\tilde{y}_T|\mathcal{Y}_{T-1}\right). \tag{16}$$

Furthermore, we know that

$$\tilde{y}_t|\mathcal{Y}_{t-1}\sim N\left(J_tZa_{t|t-1},\Omega_t\right) \tag{17}$$

where

$$\begin{aligned}\Omega_t &= E\left[\left(J_tZ\left(\alpha_t-a_{t|t-1}\right)+\tilde{\varepsilon}_t\right)\left(J_tZ\left(\alpha_t-a_{t|t-1}\right)+\tilde{\varepsilon}_t\right)'\right]\\ &= J_tZV_{t|t-1}Z'J_t'+I_{N_t}\otimes\Sigma_t.\end{aligned}$$

Consequently, we may write the log likelihood of the observed sample as

$$\ln L=-\frac{\sum_{t=1}^T N_t}{2}\ln\left(2\pi\right)-\frac{1}{2}\sum_{t=1}^T\left[\ln\left|\Omega_t\right|+\left(\tilde{y}_t-J_tZa_{t|t-1}\right)'\Omega_t^{-1}\left(\tilde{y}_t-J_tZa_{t|t-1}\right)\right]. \tag{18}$$

Due to the high dimension of Ω_t, calculation of $|\Omega_t|$ by direct calculations is extremely time consuming, and will not work on most computer systems. However, as shown in the appendix, a factorization is possible. First of all, we may rewrite $|\Omega_t|$ as

$$\left|\Omega_t\right|=\left|\Sigma_t\right|^{N_t-G}\prod_{h=1}^G\left|\Lambda_h\right| \tag{19}$$

where

$$\Lambda_h:=\begin{cases}N_1^gJ_1ZV_{t|t-1}Z'J_1'+\Sigma_t & \text{if } h=1\\ N_{h+1}^gJ_hZ\left[V_{t|t-1}^{-1}+\sum_{i=1}^{h-1}N_i^gZ'J_i'\Sigma_t^{-1}J_iZ\right]^{-1}Z'J_h'+\Sigma_t & \text{if } h>1.\end{cases}$$

Furthermore, the appendix shows that

$$\begin{aligned}\Psi_t \quad &: = \left(\tilde{y}_t-J_tZa_{t|t-1}\right)'\Omega_t\left(\tilde{y}_t-J_tZa_{t|t-1}\right)\\ &= \sum_{h=1}^G\operatorname{tr}\left[N_{ht}^g\Sigma^{-1}\operatorname{Cov}_{ht}y_{it}\right]\\ &\quad+\left(\bar{y}_t^G-Za_{t|t-1}\right)'\Xi_t\left\{I_{Gm}-Z\left[V_{t|t-1}^{-1}+Z'\Xi_tZ\right]^{-1}Z'\Xi_t\right\}\left(\bar{y}_t^G-Za_{t|t-1}\right).\end{aligned} \tag{20}$$

where N_{ht}^g is the number of members of group h at data t, $Cov_{ht}\left(y_{it}\right)$ denotes the intra-group empirical variance-covariance matrix of the y_{it}s at date t without degrees of freedom-adjustment, and $\Xi_t=\mathcal{N}^G\otimes\Sigma_t^{-1}$. ¿From equations (19) and (20) we can then calculate the likelihood value

$$\ln L=-\frac{\sum_{t=1}^T N_t}{2}\ln\left(2\pi\right)-\frac{1}{2}\sum_{t=1}^T\left[\ln\left|\Omega_t\right|+\Psi_t\right]. \tag{21}$$

An analytical solution to the ML-problem is clearly not available, although it might be possible to concentrate it with regard to the Σ_t's. We will then have to use a numerical optimization algorithm. Analytical derivatives are tedious to obtain, so it is probably desirable to rely on numerical derivatives in most applications. Since the likelihood function is often quite ill-conditioned far from the optimum, my experience has been that it is useful to use robust algorithm, for instance the Simplex algorithm, initially, and then switch to the more robust BFGS algorithm then the former starts converging. If one has a good initial point, it is probably possible to go directly to BFGS.

An alternative approach, which is very robust although somewhat slow, is the EM-algorithm developed by Dempster et al. (1977), introduced to the estimation of state space models by Engle and Watson (1983) and Shumway and Stoffer (1982). In some cases, this algorithm is superior to Simplex initially, but it should be supplemented with a more efficient algorithm when it starts converging. The idea of the EM-algorithm is to treat $\mathcal{A}_T \equiv (\alpha_1', \dots . \alpha_T')$ as missing data. From an initial estimate Θ^0 of the hyper-parameters, we can use the Kalman smoother to obtain estimates of the latent $\mathcal{A}_T$. Instead of considering the ordinary likelihood function, the EM-algorithm employs the joint likelihood function, which for model (9) is

$$
\begin{aligned}
L\left(\mathcal{Y}_T, \mathcal{A}_T; \Theta\right) = & -\frac{\sum_{t=1}^T N_t}{2} \ln\left(2\pi\right) - \frac{\sum_t N_t}{2} \ln\left|\Sigma\right| \\
& - \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \left(y_{it} - J_{g(i)t} Z \alpha_t\right)' \Sigma_t^{-1} \left(y_{it} - J_{g(i)t} Z \alpha_t\right) \\
& - \frac{\sum_t N_t}{2} \ln\left|Q\right| - \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \left(\alpha_t - F\alpha_{t-1}\right)' Q^{-1} \left(\alpha_t - F\alpha_{t-1}\right) \\
& - \frac{1}{2} \ln\left|Q_0\right| - \frac{1}{2} \left(\alpha_0 - a_0\right)' Q^{-1} \left(\alpha_0 - a_0\right).
\end{aligned}
\tag{22}
$$

Having obtained estimates of $\mathcal{A}_t$ from an estimate Θ^j,the next step in the algorithm is to maximize the expected joint likelihood function with regard to Θ. In this case, we get

$$
\begin{aligned}
& E\left[L\left(\mathcal{Y}_T, \mathcal{A}_T; \Theta\right) \left|\Theta^i\right.\right] \propto \\
& - \frac{\sum_t N_t}{2} \ln\left|\Sigma\right| - \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \operatorname{tr}\left\{\Sigma_t^{-1} \left[\left(y_{it} - J_{g(i)t} Z a^j_{t|T}\right) \left(y_{it} - J_{g(i)t} Z a^j_{t|T}\right)' + J_{g(i)t} Z V^j_{t|T} Z' J'_{g(i)t}\right]\right\} \\
& - \frac{\sum_t N_t}{2} \ln\left|Q\right| - \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \operatorname{tr}\left\{Q^{-1} \left[\left(a^j_{t|T} - F a^j_{t-1|T}\right) \left(a^j_{t|T} - F a^j_{t-1|T}\right)'\right]\right\} \\
& + V^j_{t|T} + F V^j_{t-1|T} F' - F B^j_t V^j_{t|T} - V^j_{t|T} B^j_t F'\Big] \\
& - \frac{1}{2} \ln\left|Q_0\right| - \frac{1}{2} \operatorname{tr}\left\{Q^{-1} \left[\left(a_0 - a^j_{0|T}\right) \left(a_0 - a^j_{0|T}\right)' + V^j_{0|T}\right]\right\}
\end{aligned}
\tag{23}
$$

where $B_t^j = V^j_{t-1|t-1} F' V^{j-1}_{t|t-1}$ and the parameters with superscript j are estimates from the Kalman smoother conditional on Θ^j, the hyper-parameters from the j'th iteration of the EM-algorithm. Calculating the first order conditions and simplifying, we obtain a new set of parameters Θ^{j+1}:

$$
\begin{aligned}
\Sigma_t^{j+1} & = \frac{1}{N_t} \sum_{i=1}^{N_t} \left[\left(y_{it} - J_{g(i)t} Z a^j_{t|T}\right) \left(y_{it} - J_{g(i)t} Z a^j_{t|T}\right)' + J_{g(i)t} Z V^j_{t|T} Z' J'_{g(i)t}\right] \\
& = \sum_{g=1}^{G} \frac{N_g^g}{N_t} \left[\operatorname*{Cov}_{gt}\left(y_{it}\right) + \left(\bar{y}_{gt} - J_{g(i)t} Z a^j_{t|T}\right) \left(\bar{y}_t - J_{g(i)t} Z a^j_{t|T}\right)' + J_{gt} Z V^j_{t|T} Z' J'_{gt}\right]
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
Q^{j+1} &= \frac{1}{\sum_t N_t} \sum_{t=1}^{T} N_t \left[\left(a^j_{t|T} - F a^j_{t-1|T} \right) \left(a^j_{t|T} - F a^j_{t-1|T} \right)' \right. \\
&\quad \left. + V^j_{t|T} + F V^j_{t-1|T} F' - F B^j_t V^j_{t|T} - V^j_{t|T} B^j_t F' \right] \qquad (25) \\
a_0^{j+1} &= a^j_{0|T} \qquad Q_0^{j+1} = V^j_{0|T} \qquad (26)
\end{aligned}
$$

If Σ is time-invariant, an obvious estimator is

$$
\Sigma^{j+1} = \frac{1}{\sum_{t=1}^{T} N_t} \sum_{t=1}^{T} N_t \Sigma_t^{j+1}.
$$

We can then go on to calculate a new estimate of $\mathcal{A}_t$, a new expression for the expected joint likelihood value from (23), and then calculate new estimates of the hyper-parameters from (24-26). As shown by Dempster at al. (1977), each step in this iteration will increase the likelihood value, and the estimated hyper-parameters will converge towards a local maximum of the likelihood function.

It is clear that consistent estimates of a_0 and Q_0 are not available since we do not gain further information on these parameters from a longer time series. Also, it seems that Q_0 is not well identified since it tends towards zero in most applications of the algorithm. Following Shumway and Stoffer (1982: 257), it is then probably advisable to choose a reasonable value for Q_0 rather than trying to estimate it.

5 Conclusion

I have presented a modified Kalman filtering algorithm to perform calculations on repeated samples by taking into account the particular structure of such data. The procedure makes it possible to obtain efficient estimates of underlying estimates of the laws of motion of the parameters of interest. By using the Kalman filter to smooth the estimates from each sample, we get more precise estimates in each period. Hence even if each survey is small, we get reliable estimates, so we can produce estimates with higher frequency than what has been possible so far. By defining each group as a geographical area, the procedure is also applicable for small area estimation. Finally, forecasting is simple to perform and have well-known properties when using techniques based on the Kalman filter. At the present stage, the method only admits estimation of sample means. An interesting extension would be to allow for estimation of repeated regression coefficients as in Wangen and Aasness (2002), but by integrating the estimation of the regressions with the Kalman filter.

References

Binder, D. A. and J. P. Dick (1989): Modelling and estimation for repeated surveys. *Survey Methodology* 15: 29-45.

Blight, B. J. N. and A. J. Scott (1973): A stochastic model for repeated surveys. *Journal of the Royal Statistical Society* Series B, 35: 61-66.

Dempster, A. P., N. Laird and D. B. Rubin (1977): Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society B39*, 1-38.

Doornik, J. A. (1999): *Object oriented Matrix Programming using Ox.* London: Timberlake Consultants.

Fahrmeir, L. and G. Tutz (1994): *Multivariate statistical modelling based on generalized linear models.* New York: Springer.

Gurney, M. and J. F. Daly (1965): A multivariate approach to estimation in periodic sample surveys. *American Statistical Association Proceedings of the Social Statistics Section* 8: 242-57.

Hamilton, J. D. (1994): *Time series analysis.* Princeton: Princeton University Press.

Harvey, A. C. (1989): *Forecasting, structural time series and the Kalman filter.* Cambridge: Cambridge University Press.

Harvey, A. C. and C.-H. Chung (1999): Estimating the underlying change in UK unemployment. *Journal of the Royal Statistical Society Series A* 163: 303-39.

Jessen, R. J. (1942): Statistical investigation of a farm survey for obtaining farm facts. *Iowa Agricultural Experimental Station Research Bulletin* 304: 54-59.

Lütkepohl, H. (1996): *Handbook of Matrices.* Chichester: Wiley.

Pfeffermann, D. (1991): Estimation and seasonal adjustment of population means using data from repeated surveys. *Journal of Business and Economic Statistics 9*, 163-77.

Scott, A. J. and T. M. F. Smith (1974): Analysis of repeated surveys using time series methods. *Journal of the American Statistical Association* 69: 647-78.

Shumway, R. H. and D. S. Stoffer (1982): An approach to time series smoothing and forecasting from the EM algorithm. *Journal of Time Series Analysis 3*, 253-64.

Tam, S. M. (1987): Analysis of repeated surveys using a dynamic linear model. *International Statistical Review* 55: 63-73.

Wangen, K. R. and J. Aasness (2002): Demand for manufactured and hand rolling cigarettes: A time series analysis of cross section elasticities. In K. R. Wangen: "Patterns in household tobacco consumption," Dissertation for the dr. polit. degree, University of Oslo.

Watson, M. W. and R. F. Engle (1983): Alternative algorithms for the estimation of dynamic factor, MIMIC and varying coefficient regression models. *Journal of Econometrics 23*, 385-400.

A Proofs

A.1 Proof of equation (11)

From the matrix inversion lemma (Lütkepohl 1996: 29), we have

$$\begin{aligned}
& \left(J_t Z V_{t|t-1} Z' J_t' + I_{N_t} \otimes \Sigma_t\right)^{-1} \\
= \;& I_{N_t} \otimes \Sigma_t^{-1} - I_{N_t} \otimes \Sigma_t^{-1} J_t Z \left(V_{t|t-1}^{-1} + Z' J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) J_t Z\right)^{-1} Z' J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right).
\end{aligned} \tag{27}$$

Furthermore,

$$\begin{aligned}
J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) J_t &= \begin{pmatrix} J_{g(1)t}' & \cdots & J_{g(N_t)t}' \end{pmatrix} \begin{pmatrix} \Sigma^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Sigma^{-1} \end{pmatrix} \begin{pmatrix} J_{g(1)t} \\ \vdots \\ J_{g(N_t)t} \end{pmatrix} \\
&= \sum_{i=1}^{N_t} J_{g(i)t}' \Sigma^{-1} J_{g(i)t},
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
J_{g(i)t}' \Sigma^{-1} J_{g(i)t} &= \begin{pmatrix} \mathbf{0}_{m\times m} \\ \vdots \\ I_m \\ \vdots \\ \mathbf{0}_{m\times m} \end{pmatrix} \Sigma^{-1} \begin{pmatrix} \mathbf{0}_{m\times m} & \cdots & I_m & \cdots & \mathbf{0}_{m\times m} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0}_{m\times m} & \cdots & \Sigma^{-1} & \cdots & \mathbf{0}_{m\times m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} \end{pmatrix}
\end{aligned} \tag{29}$$

where the Σ^{-1} is in the $g(i) \times g(i)$'th position. Let N_h^g denote the number of members in group h, and let $N^G = \mathrm{diag}\left(N_1^g, \ldots, N_G^g\right)$. Then

$$J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) J_t = \mathcal{N}^G \otimes \Sigma^{-1}. \tag{30}$$

Hence the Kalman gain may be written as

$$\begin{aligned}
K_t &= V_{t|t-1} Z' J_t' \left[I_{N_t} \otimes \Sigma^{-1} - \left(I_{N_t} \otimes \Sigma^{-1}\right) J_t Z \left(V_{t|t-1}^{-1} + Z' \left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right)^{-1} Z' J_t' \left(I_{N_t} \otimes \Sigma^{-1}\right)\right] \\
&= V_{t|t-1} \left[I_n - Z' \left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z \left(V_{t|t-1}^{-1} + Z' \left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right)^{-1}\right] Z' J_t' \left(I_{N_t} \otimes \Sigma^{-1}\right) \\
&= \left(V_{t|t-1}^{-1} + Z' \left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right)^{-1} Z' J_t' \left(I_{N_t} \otimes \Sigma^{-1}\right),
\end{aligned} \tag{31}$$

and then

$$a_{t|t} - a_{t|t-1} = \left(V_{t|t-1}^{-1} + Z' \left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right)^{-1} Z' J_t' \left(I_{N_t} \otimes \Sigma^{-1}\right) \left(\tilde{y}_t - J_t Z a_{t|t-1}\right). \tag{32}$$

Since

$$J_{g(i)t}' \Sigma^{-1} \left(y_{it} - J_{g(i)t} Z a_{t|t-1}\right) = \begin{pmatrix} \mathbf{0}_{m\times 1} \\ \vdots \\ y_{it} - J_{g(i)t} Z a_{t|t-1} \\ \vdots \\ \mathbf{0}_{m\times 1} \end{pmatrix}, \tag{33}$$

where the $y_{it} - J_{g(i)t} Z a_{t|t-1}$ is in the $g(i)$'th position, we have

$$\begin{aligned}
J_t' \left(I_{N_t} \otimes \Sigma^{-1}\right) \left(\tilde{y}_t - J_t Z a_{t|t-1}\right) &= \sum_{i=1}^{N_t} J_{g(i)t}' \Sigma^{-1} \left(y_{it} - J_{g(i)t} Z a_{t|t-1}\right) \\
&= \left(\mathcal{N}^G \otimes \Sigma^{-1}\right) \left(\bar{y}_t^G - Z a_{t|t-1}\right)
\end{aligned} \tag{34}$$

where

$$\bar{y}_t^G \equiv \begin{pmatrix} \frac{1}{N_1^g} \sum_{g(i)=1} y_{it} \\ \vdots \\ \frac{1}{N_G^g} \sum_{g(i)=G} y_{it} \end{pmatrix} \tag{35}$$

is the vector of stacked averages and we used the fact that $(J'_{1t},\dots J'_{Gt})' = I_{Gm}$. Consequently, the Kalman updating becomes

$$a_{t|t} = a_{t|t-1} + \left(V_{t|t-1}^{-1} + Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right)^{-1} Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right)\left(\bar{y}_t^G - Za_{t|t-1}\right), \tag{36}$$

which is only a function of group averages, and where the matrix to be inverted is of dimension $n \times n$. The expression for updating the covariance simplifies to

$$\begin{aligned} V_{t|t} &= V_{t|t-1} - \left(V_{t|t-1}^{-1} + Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right)^{-1} Z' J_t' \left(I_{N_t} \otimes \Sigma^{-1}\right) J_t Z V_{t|t-1} \\ &= \left[I_n - \left(V_{t|t-1}^{-1} + Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right)^{-1} Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right] V_{t|t-1} \\ &= \left(V_{t|t-1}^{-1} + Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right)^{-1}\left[V_{t|t-1}^{-1} + Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z - Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right] V_{t|t-1} \\ &= \left(V_{t|t-1}^{-1} + Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right) Z\right)^{-1}. \end{aligned} \tag{37}$$

It is seen that (36) may now be rewritten as

$$a_{t|t} = a_{t|t-1} + V_{t|t} Z'\left(\mathcal{N}^G \otimes \Sigma^{-1}\right)\left(\bar{y}_t^G - Za_{t|t-1}\right). \tag{38}$$

A.2 Proof of expressions (19) and (20)

Assume that $\tilde{y}_t$ is constructed such that the first $N_1^g m$ elements belong to group 1, the following $N_2^g m$ elements to group 2 and so on. Define for each group $h \in (1,\dots,G)$

$$J_h^g = \mathbf{1}_{N_h^g \times 1} \otimes J_h, \tag{39}$$

so that

$$E\tilde{y}_t|\mathcal{Y}_{t-1} = \begin{pmatrix} J_1^g \\ \vdots \\ J_G^g \end{pmatrix} Za_{t|t-1}.$$

Then the upper left $N_1^g m \times N_1^g m$-block of Ω_t contains the covariance of the elements from group 1; call this sub-matrix Ω_t^1. The upper left $(N_1^g + N_2^g)\, m \times (N_1^g + N_2^g)\, m$-block contains the covariance between the elements from group 1 and 2; call this sub-matrix $\Omega_t^{1:2}$. Generally, the covariance matrix of the elements belonging to group 1 to h is

$$\Omega_t^{1:h} = J_{1:h}^g Z V_{t|t-1} Z' J_{1:h}^{g\prime} + I_{\left(N_1^g+\dots+N_h^g\right)} \otimes \Sigma_t$$

where

$$J_{1:h}^g = \begin{pmatrix} J_1^g \\ \vdots \\ J_h^g \end{pmatrix}.$$

Hence for each $h \geq 1$

$$\Omega_t^{1:h+1} = \begin{pmatrix} \Omega_t^{1:h} & J_{1:h}^g Z V_{t|t-1} Z' J_{h+1}^{g\prime} \\ J_{h+1}^g Z V_{t|t-1} Z J_{1:h}^{g\prime} & J_{h+1}^g Z V_{t|t-1} Z' J_{h+}^{g\prime} + I_{N_{h+1}^g} \otimes \Sigma_t \end{pmatrix},$$

which means that

$$\left|\Omega_t^{1:h+1}\right| = \left|\Omega_g^{1:h}\right| \left| J_{h+1}^g Z V_{t|t-1} Z' J_{h+}^{g\prime} + I_{N_{h+1}^g} \otimes \Sigma_t - J_{1:h}^g Z V_{t|t-1} Z' J_{h+1}^{g\prime} \left(\Omega_t^{1:h}\right)^{-1} J_{h+1}^g Z V_{t|t-1} Z J_{1:h}^{g\prime} \right|. \tag{40}$$

Furthermore, the matrix inversion lemma yields

$$\begin{aligned} \left(\Omega_t^{1:h}\right)^{-1} = {} & I_{\left(N_1^g+\dots+N_h^g\right)} \otimes \Sigma_t^{-1} - \\ & \left(I_{\left(N_1^g+\dots+N_h^g\right)} \otimes \Sigma_t^{-1}\right) J_{1:h}^g Z \left[V^{-1} + Z' J_{1:h}^{g\prime} \left(I_{\left(N_1^g+\dots+N_h^g\right)} \otimes \Sigma_t^{-1}\right) J_{1:h}^g Z\right] \\ & \times Z' J_{1:h}^{g\prime} \left(I_{\left(N_1^g+\dots+N_h^g\right)} \otimes \Sigma_t^{-1}\right). \end{aligned}$$

Hence

$$\begin{aligned}
& J_{1:h}^{g} Z V_{t|t-1} Z' J_{h+1}^{g\prime} \Omega_p^{-1} J_{h+1}^{g} Z V_{t|t-1} Z' J_{1:h}^{g\prime} \\
= \; & J_{h+1}^{g} Z V_{t|t-1} Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z V_{t|t-1} Z' J_{h+1}^{g\prime} \\
& - J_{h+1}^{g} Z V_{t|t-1} Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} \\
& \times Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z V_{t|t-1} Z' J_{h+1}^{g\prime} \\
= \; & J_{h+1}^{g} Z V_{t|t-1} Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z \\
& \left\{ I_n - \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z \right\} V_{t|t-1} Z' J_{h+1}^{g\prime} \\
= \; & J_{h+1}^{g} Z V_{t|t-1} Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}^{g\prime}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& J_{h+1}^{g} Z V_{t|t-1} Z' J_{h+1}^{g\prime} + I_{N_{h+1}^{g}} \otimes \Sigma_t - J_{1:h}^{g} Z V_{t|t-1} Z' J_{h+1}^{g\prime} \Omega_p^{-1} J_{h+1}^{g} Z V_{t|t-1} Z' J_{1:h}^{g\prime} \\
= \; & J_{h+1}^{g} Z V_{t|t-1} Z' J_{h+1}^{g\prime} \\
& + I_{N_{h+1}^{g}} \otimes \Sigma_t - J_{h+1}^{g} Z V_{t|t-1} Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}^{g\prime} \\
= \; & J_{h+1}^{g} Z V_{t|t-1} \left\{ I_n - Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} \right\} Z' J_{h+1}^{g\prime} + I_{N_{h+1}^{g}} \otimes \Sigma_t \\
= \; & J_{h+1}^{g} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}^{g\prime} + I_{N_{h+1}^{g}} \otimes \Sigma_t.
\end{aligned}$$

It is difficult to calculate the determinant of this expression directly, but a Gauss-Jordan transformation yields

$$\begin{aligned}
& \left| J_{h+1}^{g} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}^{g\prime} + I_{N_{h+1}^{g}} \otimes \Sigma_t \right| \\
= \; & \left| \mathbf{1}_{N_{h+1}^{g} \times N_{h+1}^{g}} \otimes J_{h+1} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}' + I_{N_{h+1}^{g}} \otimes \Sigma_t \right| \\
= \; & \left| \begin{array}{c:c} J_{h+1} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}' + \Sigma_t & \mathbf{1}_{1 \times N_{h+1}^{g}-1} \otimes J_{h+1} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}' \\ \hdashline \mathbf{1}_{N_{h+1}^{g}-1 \times 1} \otimes (-\Sigma_t) & I_{N_{h+1}^{g}-1} \otimes \Sigma_t \end{array} \right| \\
= \; & \left| I_{N_{h+1}^{g}-1} \otimes \Sigma_t \right| \left| J_{h+1} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}' + \Sigma_t \right. \\
& \left. - \left(N_{h+1}^{g} - 1\right) J_{h+1} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}' \left(-\Sigma_t\right) \Sigma_t^{-1} \right| \\
= \; & \left|\Sigma_t\right|^{N_{h+1}^{g}-1} \left| N_{h+1}^{g} J_{h+1} Z \left[V_{t|t-1}^{-1} + Z' J_{1:h}^{g\prime} \left(I_p \otimes \Sigma_t^{-1}\right) J_{1:h}^{g} Z\right]^{-1} Z' J_{h+1}' + \Sigma_t \right|.
\end{aligned}$$

Substituting into (40), we get

$$\left|\Omega_t^{1:h+1}\right| = \left|\Omega_t^{1:h}\right| \left|\Sigma_t\right|^{N_{h+1}^{g}-1} \left| N_{h+1}^{g} J_{h+1} Z \left[V_{t|t-1}^{-1} + \sum_{i=1}^{h} N_i^{g} Z' J_i' \Sigma_t^{-1} J_i Z\right]^{-1} Z' J_{h+1}' + \Sigma_t \right|. \tag{41}$$

Furthermore,

$$\begin{aligned}
\left|\Omega_t^{11}\right| &= \left| \mathbf{1}_{N_1^{g} \times N_1^{g}} \otimes J_1 Z V_{t|t-1} Z' J_1' + I_{N_1^{g}} \otimes \Sigma_t \right| \\
&= \left| \begin{array}{cc} J_1 Z V_{t|t-1} Z' J_1' + \Sigma_t & \mathbf{1}_{1 \times N_1^{g}-1} \otimes J_1 Z V_{t|t-1} Z' J_1' \\ \mathbf{1}_{N_1^{g}-1 \times 1} \otimes (-\Sigma_t) & I_{N_1^{g}-1} \otimes \Sigma_t \end{array} \right| \\
&= \left|\Sigma_t\right|^{N_1^{g}-1} \left| N_1^{g} J_1 Z V_{t|t-1} Z' J_1' + \Sigma_t \right|.
\end{aligned} \tag{42}$$

Consequently, we may rewrite $|\Omega_t|$ as

$$|\Omega_t| = |\Sigma_t|^{N_t - G} \left| N_1^{g} J_1 Z V_{t|t-1} Z' J_1' + \Sigma_t \right| \prod_{h=2}^{G} \left| N_{h+1}^{g} J_h Z \left[V_{t|t-1}^{-1} + \sum_{i=1}^{h-1} N_i^{g} Z' J_i' \Sigma_t^{-1} J_i Z\right]^{-1} Z' J_h' + \Sigma_t \right|, \tag{43}$$

which is clearly a tractable expression.

Next, we want to simplify the expression for Ψ_t. Using the result from (27), we get

$$\begin{aligned}\Psi_t &= \left(\tilde{y}_t - J_t Z a_{t|t-1}\right)' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) \left(\tilde{y}_t - J_t Z a_{t|t-1}\right) \\ &\quad - \left(\tilde{y}_t - J_t Z a_{t|t-1}\right)' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) J_t Z \left[V_{t|t-1}^{-1} + Z' J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) J_t Z\right]^{-1} \\ &\quad \times Z' J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) \left(\tilde{y}_t - J_t Z a_{t|t-1}\right).\end{aligned}$$

Furthermore, $y_{it} - J_{g(i)} Z a_{t|t-1} = \left(y_{it} - \bar{y}_{g(i)t}\right) + \left(\bar{y}_{g(i)t} - J_{g(i)} Z a_{t|t-1}\right)$ where $\bar{y}_{gt}$ is the average value of y in group g at date t. Hence

$$\begin{aligned}&\left(\tilde{y}_t - J_t Z a_{t|t-1}\right)' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) \left(\tilde{y}_t - J_t Z a_{t|t-1}\right) \\ =& \sum_{i=1}^{N_t} \left[\left(y_{it} - \bar{y}_{g(i)t}\right)' \Sigma_t^{-1} \left(yt - \bar{y}_{g(i)t}\right) + \left(\bar{y}_{g(i)t} - J_{g(i)} Z a_{t|t-1}\right)' \Sigma^{-1} \left(\bar{y}_{g(i)t} - J_{g(i)} Z a_{t|t-1}\right)\right] \\ =& \sum_{g=1}^{G} \operatorname{tr}\left[N_g^g \Sigma^{-1} \underset{gt}{\operatorname{Cov}}\, y_{it}\right] + \left(\bar{y}_t^G - Z a_{t|t-1}\right)' \left(\mathcal{N}^G \otimes \Sigma_t^{-1}\right) \left(\bar{y}_t^G - Z a_{t|t-1}\right)\end{aligned}$$

where the last line uses the fact that the trace of a scalar is the scalar. ¿From (34) it follows that

$$\begin{aligned}&\left(\tilde{y}_t - J_t Z a_{t|t-1}\right)' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) J_t Z \left[V_{t|t-1}^{-1} + Z' J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) J_t Z\right]^{-1} \\ &\times Z' J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) \left(\tilde{y}_t - J_t Z a_{t|t-1}\right) \\ =& \left(\bar{y}_t^G - Z a_{t|t-1}\right)' \left(\mathcal{N}^G \otimes \Sigma_t^{-1}\right) Z \left[V_{t|t-1}^{-1} + Z' J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) J_t Z\right]^{-1} \\ &\times Z' \left(\mathcal{N}^G \otimes \Sigma_t^{-1}\right) \left(\bar{y}_t^G - Z a_{t|t-1}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\Psi_t =& \sum_{g=1}^{G} \operatorname{tr}\left[N_g^g \Sigma^{-1} \underset{gt}{\operatorname{Cov}}\, y_{it}\right] \\ &+ \left(\bar{y}_t^G - Z a_{t|t-1}\right)' \left(\mathcal{N}^G \otimes \Sigma_t^{-1}\right) \left\{I_{Gm} - Z \left[V_{t|t-1}^{-1} + Z' J_t' \left(I_{N_t} \otimes \Sigma_t^{-1}\right) J_t Z\right]^{-1} \right. \qquad (44) \\ &\left. \times\, Z' \left(\mathcal{N}^G \otimes \Sigma_t^{-1}\right)\right\} \left(\bar{y}_t^G - Z a_{t|t-1}\right) \\ =& \sum_{g=1}^{G} \operatorname{tr}\left[N_g^g \Sigma^{-1} \underset{gt}{\operatorname{Cov}}\, y_{it}\right] \\ &+ \left(\bar{y}_t^G - Z a_{t|t-1}\right)' \left(\mathcal{N}^G \otimes \Sigma_t^{-1}\right) \left\{I_{Gm} - Z \left[V_{t|t-1}^{-1} + Z' \left(\mathcal{N}^G \otimes \Sigma_t^{-1}\right) Z\right]^{-1} \right. \qquad (45) \\ &\left. \times\, Z' \left(\mathcal{N}^G \otimes \Sigma_t^{-1}\right)\right\} \left(\bar{y}_t^G - Z a_{t|t-1}\right).\end{aligned}$$

B A computer program

Below I give the main routines of a computer program to implement the algorithm described above written in the programming language Ox (see Doornik (1999) for a description). The procedure `filter` implements the Kalman filter for surveys as described in Section 2. The procedure `smooth` is the associated Kalman smoother. Finally, the procedure `loglikelihood` returns the log likelihood of the model and is used for maximum likelihood estimation. The full program is available from the author upon request.

```
filter(const model,const a0, const Q0, const sigma, const Q, const data,
       const a_pred_out, const V_pred_out, const a_filter_out, const V_filter_out)
{ decl G=model[0], T=model[1], m=model[2], n=model[3], F=model[4], Z=model[5],
       y=data[0], Cov=data[1], N=data[2],
       a_pred=array(M_NAN),
       V_pred=array(M_NAN),
       a_filter=array(a0),
```

```
       V_filter=array(Q0),
       t,NtS;

  for (t=1; t<=T; ++t)
  { NtS=diag(N[t])**invertsym(sigma);
          //Matrix with N_t along diagonal ** Sigma^-1
    a_pred  =a_pred  |(F*a_filter[t-1]);
    V_pred  =V_pred  |(F*V_filter[t-1]*F'+Q);
    V_filter=V_filter|invertsym(invertsym(V_pred[t])+Z'NtS*Z);
    a_filter=a_filter|(a_pred[t]+V_filter[t]*Z'NtS*(y[t]-Z*a_pred[t]));
  }

  if (a_pred_out)   a_pred_out[0]  =a_pred;
  if (V_pred_out)   V_pred_out[0]  =V_pred;
  if (a_filter_out) a_filter_out[0]=a_filter;
  if (V_filter_out) V_filter_out[0]=V_filter;
}


smooth(const model, const a_pred, const V_pred, const a_filter, const V_filter,
       const a_smooth_out, const V_smooth_out, const B_out)
{ decl G=model[0], T=model[1], m=model[2], n=model[3], F=model[4], Z=model[5],
       a_smooth=new array[T+1],
       V_smooth=new array[T+1],
       B       =new array[T+1],
       t;

  a_smooth[T]=a_filter[T];
  V_smooth[T]=V_filter[T];
  for (t=T; t>0; --t)
  { B[t]=V_filter[t-1]*F'invertsym(V_pred[t]);
    a_smooth[t-1]=a_filter[t-1]+B[t]*(a_smooth[t]-a_pred[t]);
    V_smooth[t-1]=V_filter[t-1]+B[t]*(V_smooth[t]-V_pred[t])*B[t]';
  }

  if (a_smooth_out) a_smooth_out[0]=a_smooth;
  if (V_smooth_out) V_smooth_out[0]=V_smooth;
  if (B_out)        B_out[0]       =B;

}

Jg(const g, const G, const m)
// Returns m*Gm matrix with unit matrix in g'th postion
{ if (G==1) return unit(m);
  if (g==1) return (unit(m)~(zeros(m,(G-1)*m)));
  if (g==G) return ((zeros(m,(G-1)*m))~unit(m));
  return (zeros(m,(g-1)*m)~unit(m)~zeros(m,(G-g)*m));
}


lndet(const A)
// More convenient form of logdet
```

```
{ decl asign;
  return logdet(A,&asign);
}


loglikelihood(const model, const a0, const Q0, const sigma, const Q, const data)
{ decl G=model[0], T=model[1], m=model[2], n=model[3], F=model[4], Z=model[5],
       y=data[0], Cov=data[1], N=data[2],
       a, V,   // Predicted values
       t,g,Omega_t, Psi_t, ll=0,sumZJSJZ,Nt,yG,NtS,
       inv_sigma=invertsym(sigma);

  filter(model,a0,Q0,sigma,Q,data,&a,&V,0,0);

  for (t=1;t<=T;++t)
  { NtS=diag(N[t])**invertsym(sigma);
           //Matrix with N_t along diagonal ** Sigma^-1
    g=1;
    Omega_t=lndet(N[t][g-1]*Jg(g,G,m)*Z*V[t]*Z'Jg(g,G,m)+sigma);
    Psi_t=N[t][g-1]*trace(inv_sigma*Cov[t][g-1]);
    sumZJSJZ=N[t][g-1]*Z'Jg(g,G,m)'*inv_sigma*Jg(g,G,m)*Z;
    Nt=N[t][g-1];
    yG=y[t][g-1];

    for (g=2; g<=G; ++g)
    { Omega_t+=lndet(N[t][g-1]*Jg(g,G,m)*Z*
             invertsym(invertsym(V[t])+sumZJSJZ)*Z'Jg(g,G,m)+sigma);
      Psi_t+=N[t][g-1]*trace(inv_sigma*Cov[t][g-1]);
      sumZJSJZ+=N[t][g-1]*Z'Jg(g,G,m)'*inv_sigma*Jg(g,G,m)*Z;
      Nt+=N[t][g-1];
      yG|=y[t][g-1];
    }
    Omega_t+=(Nt-G)*lndet(sigma);
    Psi_t+=(yG-Z*a[t])'*NtS*(unit(G*m)-Z*invertsym(invertsym(V[t])+Z'NtS*Z)*Z'NtS)
             *(yG-Z*a[t]);
    ll-=0.5*(Omega_t+Psi_t);
  }
  return ll;
}
```

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