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Thomas von Brasch, Johan Byström and
Lars Petter Lystad

Optimal Control and the Fibonacci Sequence

Abstract:
We bridge mathematical number theory with that of optimal control and show that a generalised Fibonacci sequence enters the control function of finite horizon dynamic optimisation problems with one state and one control variable. In particular, we show that the recursive expression describing the first-order approximation of the control function can be written in terms of a generalised Fibonacci sequence when restricting the final state to equal the steady state of the system. Further, by deriving the solution to this sequence, we are able to write the first-order approximation of optimal control explicitly. Our procedure is illustrated in an example often referred to as the Brock-Mirman economic growth model.

Keywords: Fibonacci sequence, Golden ratio, Mathematical number theory, Optimal control.

AMS classification: 11B39, 93C55, 37N40.

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Address: Thomas von Brasch, Statistics Norway, Research Department. E-mail: thomas.vonbrasch@ssb.no

Johan Byström, Luleå University of Technology: Department of Mathematics. E-mail: johanb@ltu.se

Lars Petter Lystad, Narvik University College: Department of Technology. E-mail: lpl@hin.no

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Sammendrag

Vi bygger en bro mellom tallteori og optimal kontrollteori ved å vise at en generalisert Fibonacci-rekke inngår i kontrollfunksjonen tilhørende et generelt dynamisk optimeringsproblem, formulert over endelig tid og med én kontroll og én tilstandsvariabel. Det blir vist at den lineære approksimasjonen av kontrollfunksjonen kan skrives ut ifra den generaliserte Fibonacci-rekken når man betinger tilstandsvariabelen i siste periode til å nå systemets likevekt. Ved å utlede løsningen til den generelle Fibonacci-rekken kan den lineære approksimasjonen av den optimale kontrollfunksjonen skrives eksplisitt. Det hele illustreres i et eksempel fra økonomisk teori som ofte blir omtalt som Brock-Mirman modellen.

1 Introduction

Approximating optimal control problems has a long history and dates at least back to McReynolds [1]. Lystad [2] and Magill [3, 4] are early applications of the first-order approximation within economics. A good account of how the technique has been used in economics can be found in Judd [5]. Recently, this method of approximation has been extended to also handle stochastic rational expectation models with forward looking variables, see, e.g., Levine et al. [6] and Benigno and Woodford [7].

The field of bridging optimal control and number theory via the Fibonacci sequence is relatively new. Benavoli et al. [8] show the relationship between the Fibonacci sequence and the Kalman filter with a simple structure of the plant model. Capponi et al. [9] derive a similar result in a continuous time setting. Donoghue [10] show a linkage between the Kalman filter, the linear quadratic control problem and a Fibonacci system defined by adding a control input to the recursion relation generating the Fibonacci numbers. Byström et al. [11] derive a relationship between linear quadratic problems and a generalised Fibonacci sequence. We build upon and extend these results for control problems in a generalised form.

The main contribution of this article is to bridge the area of mathematical number theory with that of optimal control. This is done by using a generalised Fibonacci sequence for solving finite horizon dynamic optimisation problems with one state and one control variable. The solution method proposed reveals important properties of the optimal control problem. In particular, we show how the first-order approximation of the optimal control function can be written in terms of these generalised Fibonacci numbers. Further, by developing the explicit solution of the generalised Fibonacci sequence, we are able to provide a non-recursive solution of the first-order approximation.

The structure of the paper is as follows. In Section 2, the optimal control problem is defined and the expression describing the first-order approximation of the optimal control is stated. In Section 3, we contribute to the literature by developing the linkage between the optimal control function and the Fibonacci sequence. To illustrate our procedure, we show how the method can be applied to the Brock-Mirman economic growth model. We derive explicit solutions to the generalised Fibonacci numbers in Section 4, which further enables us to write the first-order approximation of optimal control explicitly. The last section contains a summary and concluding remarks.

2 The Optimal Control Problem

The deterministic optimal control problem consists of minimising an objective function subject to the process describing the evolution of the state variable, given a restriction on the terminal state variable.[1] For $0 \leq t \leq$

[1] The optimal control problem has been widely used within the field of economics, see e.g., Ljungqvist and Sargent [12].

$T-1$, we define the objective function

$$\sum_{t=0}^{T-1} \beta^t f(x_t, u_t), \tag{1}$$

where $0 < \beta \leq 1$ is a discount factor and where $x_t \in \mathbb{R}$ represents a state variable and $u_t \in \mathbb{R}$ denotes the control variable. Further, it is assumed that standard regularity conditions hold, i.e., the criterion function f is sufficiently smooth and convex and policies, that are feasible, lie within a compact and convex set. The evolution of the state variable is described by the discrete time system

$$x_{t+1} := Ax_t + Bu_t, \qquad t = 0, 1, ..., T-1, \tag{2}$$

for a given initial condition x_0. We assume there exists a control, which ensures that the state never changes. We refer to such a control as a steady state control and denote it $(\bar{u})$ and, correspondingly, we denote the steady state $(\bar{x})$. The final state of the discrete time system (2) is restricted to be the steady state

$$x_T = \bar{x}. \tag{3}$$

From this it follows that a steady state is characterised by two properties. First, the state is constant and thus time invariant. Second, the steady state control is optimal, i.e., given that the system starts out at the steady state, it is optimal to remain at the steady state through all time periods. The assumption that there exists a steady state is necessary in order to use the generalised Fibonacci sequence to write the first-order approximation of optimal control explicitly.

The optimal control problem is therefore the problem of minimising the objective function (1) subject to both the transition function (2) and the fixed final state (3).

Even though the optimal control problem is deterministic, the approach used in this article can be generalised to handle stochastic control problems; see, e.g., Levine et al. [6] and Benigno and Woodford [7].

In general, it is not possible to find an explicit expression describing the optimal control function. However, it may be possible to find a recursive expression describing the first-order approximation of the optimal control. In the following well known result, we let the second partial derivatives of the criterion function f, evaluated at the steady state, be denoted by $f_{\bar{x}\bar{x}} := \frac{\partial^2 f}{\partial \bar{x} \partial \bar{x}}$, $f_{\bar{x}\bar{u}} := \frac{\partial^2 f}{\partial \bar{x} \partial \bar{u}}$ and $f_{\bar{u}\bar{u}} := \frac{\partial^2 f}{\partial \bar{u} \partial \bar{u}}$.

Theorem 2.1 *Consider the optimal control problem, i.e., minimising* (1) *subject to* (2) *and* (3)*. The first-order approximation is given by the linear control function (for* $0 \leq t \leq T-1$*)*

$$u_t = \bar{u} - (L_t^{\mathbf{a}} - L_t^{\mathbf{b}} + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})(x_t - \bar{x}), \tag{4}$$

where $L_t^{\mathbf{a}}$ *is given by the equations*

$$L_t^{\mathbf{a}} := (f_{\bar{u}\bar{u}} + \widetilde{B} S_{t+1} \widetilde{B})^{-1} (\widetilde{B} S_{t+1} \widetilde{A}) \tag{5}$$

$$S_t := \widetilde{A}^2 S_{t+1} f_{\bar{u}\bar{u}} (f_{\bar{u}\bar{u}} + \widetilde{B}^2 S_{t+1})^{-1} + \widetilde{R}, \quad S_T := 0, \tag{6}$$

where the second equation is known as the Riccati equation and where we have used the auxiliary variables $\widetilde{A} := \beta^{1/2}(A - B f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})$, $\widetilde{B} := \beta^{1/2} B$ *and* $\widetilde{R} := (f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}} f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})$*. The last part of the feedback coefficient* ($L_t^{\mathbf{b}}$) *represents the linear part which ensures that the control function will drive the state to the steady state at the final time period*

$$L_t^{\mathbf{b}} := (f_{\bar{u}\bar{u}} + \widetilde{B} S_{t+1} \widetilde{B})^{-1} \widetilde{B} W_{t+1} P_t^{-1} W_t, \tag{7}$$

where the two auxiliary variables W_t *and* P_t *are given by*

$$W_t := \left(\widetilde{A} - \widetilde{B} L_t^{\mathbf{a}}\right) W_{t+1}, \ W_T := 1, \tag{8}$$

$$P_t := P_{t+1} - W_{t+1}^2 \widetilde{B}^2 \left(f_{\bar{u}\bar{u}} + \widetilde{B}^2 S_{t+1}\right)^{-1}, \ P_T := 0. \tag{9}$$

Proof: See Section 4.5 *in Lewis and Syrmos* [13] *or Appendix* 6.2.

3 Connecting Fibonacci with Optimal Control

The Fibonacci sequence is named after the Italian mathematician Leonardo Pisano Bigollo (1170 - c. 1250), most commonly known as Leonardo Fibonacci. With his most important work, the book of number theory, Liber Abaci, he spread the Hindu-Arabic numeral system to Europe. In Liber Abaci he also introduced what many will associate him with today, the Fibonacci sequence

$$\{F_n\}_{n=0}^{\infty} = 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots.$$

This sequence is characterised by the initial values 0 and 1 and each subsequent number being the sum of the previous two. It is thus fully described by the difference equation

$$F_n := F_{n-1} + F_{n-2},$$

with initial values $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has been connected to such diverse fields as nature, art, geometry, architecture, music and even for the calculation of π, see e.g., Castellanos [14]. One of the most fascinating facts is that the ratio of two consecutive numbers ($H_n := F_{n-1}/F_n$)

$$\{H_n\} = 0, \quad 1, \quad .5, \quad .666..., \quad .600, \quad .625, \quad .615..., \quad .619..., \quad .617..., \quad .618..., \quad \ldots \tag{10}$$

converges to the inverse of the golden ratio: $\phi^{-1} := 2/(1+\sqrt{5}) \approx .618$. The golden ratio is mathematically interesting for a variety of reasons, e.g., it holds the property that its square is equal to $\phi + 1$ and its inverse is equal to the number itself minus one, i.e.,

$$\phi^{-1} = \phi - 1. \tag{11}$$

The main contribution of this article is that we are able to connect a generalised Fibonacci sequence to optimal control theory.

Definition 3.1 *(Generalised Fibonacci sequence) The generalised Fibonacci sequence is defined by the second-order difference equation*

$$\mathcal{F}_{n+2} := a\mathcal{F}_{n+1} + b_{n+2}\mathcal{F}_n, \tag{12}$$

with the constant coefficient a, the time varying coefficient b_{n+2} and with given initial values $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$.

Moreover, we define the ratio of two consecutive generalised Fibonacci numbers by

$$\mathcal{H}_n := \mathcal{F}_{n-1}/\mathcal{F}_n, \qquad n = 1, 2, \ldots$$

Theorem 3.1 *Consider the generalised Fibonacci sequence with the particular coefficients $a = \beta^{1/2}B$ and $b_{n+2} = f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\widetilde{A}^2$ when n is even and $b_{n+2} = f_{\bar{u}\bar{u}}\widetilde{R}^{-1}$ when n is odd, where we have used the auxiliary transformations $\widetilde{A} = \beta^{1/2}(A - Bf_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})$ and $\widetilde{R} = (f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}}f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})$. The first-order approximation in*

Theorem 2.1 *can then be written*

$$u_t = \bar{u} - \left(\widetilde{A}\mathcal{H}_{2(T-t)-1} + \frac{\widetilde{A}\left(\widetilde{A}f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\right)^{2(T-t-1)}}{\mathcal{F}_{2(T-t)-1}\mathcal{F}_{2(T-t)}} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}} \right)(x_t - \bar{x}).$$

Proof: See Appendix 6.5.

Corollary 3.1 *If* $\widetilde{A}^2 = 1$*, the first-order approximation of the control function simplifies to*

$$u_t = \bar{u} - \left(\widetilde{A}\mathcal{H}_{2(T-t)} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}\right)(x_t - \bar{x}).$$

Proof: See Appendix 6.6.

3.1 Example: The Brock-Mirman Economic Growth Model

Consider the standard textbook economic growth model often referred to as the Brock-Mirman model [15] [2]

$$\begin{aligned} &\min_{\{u_t\}_{t=0}^{T-1}} -\sum_{t=0}^{T-1} \beta^t \ln(\gamma x_t^\alpha - u_t) \\ s.t. \quad & x_{t+1} = u_t \quad x_0 > 0 \\ & x_T = \bar{x}. \end{aligned} \tag{13}$$

The steady state of this model is given by $\bar{x} = \bar{u} = (\alpha\beta\gamma)^{1/(1-\alpha)}$.[3] Simplifying the example we normalise the steady state to unity ($\bar{x} = \bar{u} = 1$) by imposing $\beta = 1$ and $\gamma = \alpha^{-1}$. From the transition equation (13) it follows that $A = 0$ and $B = 1$. In order to make the example particularly neat we let $\alpha = 1 - \phi^{-1}$ where ϕ is the golden ratio. It then follows from the criterion function f that[4]

$$f_{\bar{x}\bar{x}} = 2(1 - \phi^{-1}), \quad f_{\bar{u}\bar{u}} = 1 - \phi^{-1}, \quad f_{\bar{x}\bar{u}} = -(1 - \phi^{-1}),$$

which together imply $f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}} = -1$, $\widetilde{A} = 1$ and $\widetilde{R} = f_{\bar{x}\bar{x}} - f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}^2 = 1 - \phi^{-1}$. Since $\widetilde{A} = 1$, we can apply Corollary 3.1 which yields the first-order approximation of the control function

$$u_t = 1 - \left(\mathcal{H}_{2(T-t)} - 1\right)(x_t - 1). \tag{14}$$

[2] See Appendix 6.7 for some narrative details on this model.
[3] See Appendix 6.8.
[4] See Appendix 6.9.

With the above choice of parameter values the sequence $\mathcal{H}$ is in this example given by the original set of Fibonacci ratios H, see (10).[5] In Table 1 the optimal solution is compared with the control given by equation (14). At the initial time period, the discrepancy between the optimal control and the first-order approximation is 0.6 %.

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
x_t^*	0.8000	0.9183	0.9681	0.9879	0.9960	1.0000
u_t^*	0.9183	0.9681	0.9879	0.9960	1.0000	
u_t	0.9236	0.9689	0.9880	0.9960	1.0000	
$H_{2(5-t)}$	0.6182	0.6190	0.6250	0.6667	1.0000	

Table 1: Comparing the optimal control with the first-order approximation. The first and second row provide the optimal solution to the Brock-Mirman model. In the third row the Fibonacci based control is presented as given by equation (14). The sequence in the fourth row is every second element of the original set of Fibonacci ratios (10) given in reverse.

4 An Explicit Solution of the Control Function in Theorem 3.1

In order to find an explicit solution of the control function in Theorem 3.1 we observe that the undetermined expressions in the control function consist of even and odd indexed generalised Fibonacci numbers only, i.e., the sequence

$$\mathcal{H}_{2(T-t)-1} = \mathcal{F}_{2(T-t-1)}/\mathcal{F}_{2(T-t)-1},$$

has even-indexed Fibonacci numbers in the numerator and odd-indexed numbers in the denominator. The problem of finding an explicit solution of the control function is thus reduced to finding an explicit solution of the odd and even indexed Fibonacci sequence. With this goal in mind, we note that the generalised Fibonacci sequence can be written

$$\begin{aligned}\mathcal{F}_{n+2} &= a\mathcal{F}_{n+1} + b_{n+2}\mathcal{F}_n \\ &= a(a\mathcal{F}_n + b_{n+1}\mathcal{F}_{n-1}) + b_{n+2}\mathcal{F}_n \\ &= (a^2 + b_{n+2} + b_{n+1})\mathcal{F}_n - b_n b_{n+1}\mathcal{F}_{n-2}.\end{aligned}$$

[5]See Appendix 6.10.

Using the particular coefficient values $a = \widetilde{B}$ and $b_{n+2} = f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\widetilde{A}^2$ when n is even and $b_{n+2} = f_{\bar{u}\bar{u}}\widetilde{R}^{-1}$ when n is odd yields

$$\mathcal{F}_{n+2} = (\widetilde{B}^2 + f_{\bar{u}\bar{u}}\widetilde{R}^{-1}(\widetilde{A}^2 + 1))\mathcal{F}_n - f_{\bar{u}\bar{u}}^2\widetilde{R}^{-2}\widetilde{A}^2\mathcal{F}_{n-2}. \tag{15}$$

Even though the Fibonacci sequence under consideration has time varying coefficients (b_{n+2}), the sequence describing every second generalised Fibonacci number (15) has constant coefficients, see also Byström et al. [11]. Since a second-order difference equation with constant coefficients can be written in the form of (15), the solution to (15) is well known. Given the auxiliary parameters $\widetilde{R} = (f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}}f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})$, $c_1 := \sqrt{\widetilde{B}^2 + f_{\bar{u}\bar{u}}\widetilde{R}^{-1}(1-\widetilde{A})^2}$, $c_2 := f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\widetilde{A}$ and $r_{1,2} := (c_1 \pm \sqrt{c_1^2 + 4c_2})/2$, the explicit expressions for the Fibonacci sequences entering the control function are given by[6]

$$\mathcal{F}_{2(T-t-1)} = \widetilde{B}\,\frac{r_1^{2(T-t-1)} - r_2^{2(T-t-1)}}{r_1^2 - r_2^2} \tag{16}$$

$$\mathcal{F}_{2(T-t)} = \widetilde{B}\,\frac{r_1^{2(T-t)} - r_2^{2(T-t)}}{r_1^2 - r_2^2} \tag{17}$$

$$\mathcal{F}_{2(T-t)-1} = \frac{(r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1}}{r_1^2 - r_2^2}. \tag{18}$$

Inserting these expressions into Theorem 3.1 and Corollary 3.1 yield the following results:

Corollary 4.1 *The explicit solution of the control function as given in Theorem* 3.1 *is given by*

$$u_t = \bar{u} - \Bigg(\widetilde{A}\widetilde{B}\frac{r_1^{2(T-t-1)} - r_2^{2(T-t-1)}}{(r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1}} + \widetilde{A}\widetilde{B}^{-1}\frac{(r_1^2 - r_2^2)^2\left(\widetilde{A}f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\right)^{2(T-t-1)}}{\left(r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1}\right)\left(r_1^{2(T-t)} - r_2^{2(T-t)}\right)} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}\Bigg)(x_t - \bar{x}).$$

Corollary 4.2 *The explicit solution of the control function as given in Corollary* 3.1 *is given by*

$$u_t = \bar{u} - \left(\widetilde{A}\widetilde{B}^{-1}\frac{(r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1}}{r_1^{2(T-t)} - r_2^{2(T-t)}} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}\right)(x_t - \bar{x}).$$

[6]See Appendix 6.11.

4.1 Example: The Brock-Mirman Economic Growth Model Continued

The analytical solution describing the first-order approximation of the control function in the Brock-Mirman model is given directly from Corollary 4.2. Since from (65) we have $c_1 = c_2 = 1$, the roots of the characteristic equation are given by

$$r_{1,2} = \frac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

In terms of the golden ratio we can write the solution as $r_1 = \phi$ and $r_2 = 1 - \phi = -\phi^{-1}$. By inserting the relationships $\widetilde{A} = \widetilde{B} = -f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} = \bar{x} = \bar{u} = 1$ into Corollary 4.2 yields the explicit expression

$$u_t = 1 - \left(\frac{\phi^{2(T-t)-1} + \phi^{1-2(T-t)}}{\phi^{2(T-t)} + \phi^{-2(T-t)}} - 1 \right) (x_t - 1).$$

5 Conclusion

In this article we have shown how to use a generalised Fibonacci sequence for solving finite horizon dynamic optimisation problems. The solution method proposed has revealed important properties of the optimal control problem. In particular, we have shown how the first-order approximation of the optimal control function can be written in terms of these generalised Fibonacci numbers. Further, by developing the explicit solution of the generalised Fibonacci sequence we were able to provide a non-recursive solution of the first-order approximation. The procedure has been illustrated with the Brock-Mirman economic model. On a general level, we have thus bridged the area of mathematical number theory with that of optimal control.

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6 Appendix

6.1 Nomenclature

Symbols

$\sum \beta^t f$	Objective function
β	Discount factor
f	Criterion function
$\mathcal{L}$	Lagrangian
$\bar{x}$	Steady state
$\bar{u}$	Steady state control
t	Variable time index
F	Fibonacci sequence
H	Fibonacci ratio sequence
$\mathcal{F}$	Generalised Fibonacci sequence
$\mathcal{H}$	Generalised Fibonacci ratio sequence
ϕ	Golden ratio

Transformations and important relationships

ϕ	$= (1 + \sqrt{5})/2$
$d\widetilde{u}_t$	$= (du_t + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} dx_t)$
$\widetilde{R}$	$= (f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}} f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})$
$\widetilde{x}_t$	$= \beta^{t/2} dx_t$
$\widetilde{u}_t$	$= \beta^{t/2} du_t$
$\widetilde{A}$	$= \beta^{1/2}(A - B f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})$
$\widetilde{B}$	$= \beta^{1/2} B$
H_n	$= F_{n-1}/F_n$
$\mathcal{H}_n$	$= \mathcal{F}_{n-1}/\mathcal{F}_n$
c_1	$= \sqrt{\widetilde{B}^2 + f_{\bar{u}\bar{u}} \widetilde{R}^{-1} (1 - \widetilde{A})^2}$
c_2	$= f_{\bar{u}\bar{u}} \widetilde{R}^{-1} \widetilde{A}$
$r_{1,2}$	$= (c_1 \pm \sqrt{c_1^2 + 4c_2})/2$

6.2 Proof: Theorem 2.1

The Lagrangian ($\mathcal{L}$) of the optimal control problem becomes

$$\mathcal{L} := \mu_{T+1}(x_T - \bar{x}) + \sum_{t=0}^{T-1} \left(\beta^t f(x_t, u_t) + \lambda_{t+1}(Ax_t + Bu_t - x_{t+1})\right), \tag{19}$$

where μ_{T+1} and λ_{t+1} represent Lagrangian multipliers. A necessary condition for optimality, assuming that standard regularity conditions hold, i.e., the criterion function f is sufficiently smooth and convex and policies, that are feasible, lie within a compact and convex set, is that the first variation of the Lagrangian is zero. In particular, the first variation of the Lagrangian evaluated at the steady state is zero. An optimal control minimising the Lagrangian (19) can thus be approximated by an incremental control minimising the second variation

$$\begin{aligned} d^2\mathcal{L} &= d\mu_{T+1} dx_T + \frac{1}{2}\sum_{t=0}^{T-1} \Big(\beta^t (dx_t f_{\bar{x}\bar{x}}\, dx_t + du_t f_{\bar{u}\bar{u}} du_t + 2du_t f_{\bar{x}\bar{u}} dx_t) \\ &\qquad + 2d\lambda_{t+1}(Adx_t + Bdu_t - dx_{t+1}) \Big), \end{aligned}$$

where increments are made around the steady state, i.e., $du_t := u_t - \bar{u}$ and $dx_t := x_t - \bar{x}$, and where e.g., the second partial derivative of f with respect to x_t, evaluated at the steady state, is denoted by $f_{\bar{x}\bar{x}}$. This latter problem is recognised as the Lagrangian of the auxiliary discounted linear quadratic problem (DLQP)

$$\begin{aligned} \text{(DLQP)} \qquad & \min_{\{du_t\}_{t=0}^{T-1}} \frac{1}{2}\sum_{t=0}^{T-1} \beta^t \left(dx_t f_{\bar{x}\bar{x}} dx_t + du_t f_{\bar{u}\bar{u}} du_t + 2du_t f_{\bar{x}\bar{u}} dx_t\right) \\ & s.t. \quad dx_{t+1} = Adx_t + Bdu_t \qquad (20) \\ & \qquad\quad dx_T = 0, \qquad (21) \end{aligned}$$

where $d\lambda_{t+1}$ and $d\mu_{T+1}$ represent the multipliers associated with the constraints (20) and (21). In order to simplify notation, we note the following identity (assuming $f_{\bar{u}\bar{u}}^{-1}$ exists)

$$\begin{aligned} & dx_t f_{\bar{x}\bar{x}} dx_t + du_t f_{\bar{u}\bar{u}} du_t + 2du_t f_{\bar{x}\bar{u}} dx_t \\ &= dx_t (f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}} f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}}) dx_t + (du_t + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} dx_t) f_{\bar{u}\bar{u}} (du_t + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} dx_t). \end{aligned}$$

Defining $d\widetilde{u}_t := (du_t + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} dx_t)$ and $\widetilde{R} = (f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}} f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})$, the objective function in the problem (DLQP) is equivalent to

$$\frac{1}{2}\sum_{t=0}^{T-1} \beta^t \left(dx_t \widetilde{R} dx_t + d\widetilde{u}_t f_{\bar{u}\bar{u}} d\widetilde{u}_t \right). \tag{22}$$

The constraint can be altered correspondingly. Inserting $du_t = d\widetilde{u}_t - f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} dx_t$ into (20) gives

$$dx_{t+1} = (A - Bf_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}}) dx_t + Bd\widetilde{u}_t. \tag{23}$$

In order to convert the problem to one without discounting, we define the variables $\widetilde{x}_t := \beta^{t/2} dx_t$ and $\widetilde{u}_t := \beta^{t/2} d\widetilde{u}_t$. Substituting these newly defined variables into (22) and (23) yields the linear quadratic problem

$$\text{(LQP)} \qquad \min_{\{\widetilde{u}_t\}_{t=0}^{T-1}} \frac{1}{2}\sum_{t=0}^{T-1} \left(\widetilde{x}_t \widetilde{R} \widetilde{x}_t + \widetilde{u}_t f_{\bar{u}\bar{u}} \widetilde{u}_t \right)$$

$$s.t. \quad \widetilde{x}_{t+1} = \widetilde{A}\widetilde{x}_t + \widetilde{B}\widetilde{u}_t \tag{24}$$

$$\widetilde{x}_T = 0, \tag{25}$$

where $\widetilde{A} = \beta^{1/2}(A - Bf_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})$ and $\widetilde{B} = \beta^{1/2} B$. Variables with a tilde are in the problem (LQP) thus transformed from the problem (DLQP). As a result, the problem of finding the optimal plan that minimises the problem (LQP) is equivalent to finding the optimal plan which minimises the problem (DLQP) using the appropriate transformations. The problem (LQP) is well known and its solution is given by[7]

$$\widetilde{u}_t = -(L_t^{\mathbf{a}} - L_t^{\mathbf{b}})\widetilde{x}_t. \tag{26}$$

This control function describes the optimal control to the linear quadratic problem as a linear function of the state variable. The time varying coefficient in front of the state variable consists of two parts. The first part ($L_t^{\mathbf{a}}$) is the feedback equation of a linear quadratic problem when there is no restriction on the final state, i.e., $\widetilde{x}_T$ is free to vary. It is determined by the equations

$$L_t^{\mathbf{a}} = (f_{\bar{u}\bar{u}} + \widetilde{B} S_{t+1} \widetilde{B})^{-1} (\widetilde{B} S_{t+1} \widetilde{A}) \tag{27}$$

$$S_t = \widetilde{A}^2 S_{t+1} f_{\bar{u}\bar{u}} (f_{\bar{u}\bar{u}} + \widetilde{B}^2 S_{t+1})^{-1} + \widetilde{R}, \quad S_T = 0, \tag{28}$$

[7]See Appendix 6.3. For a textbook derivation see Section 4.5 in Lewis and Syrmos [13].

where the second equation is known as the Riccati equation. The last part of the feedback coefficient ($L_t^{\mathbf{b}}$) represents the linear part which ensures that the control function will drive the state to zero at the final time period

$$L_t^{\mathbf{b}} = (f_{\bar{u}\bar{u}} + \widetilde{B}S_{t+1}\widetilde{B})^{-1}\widetilde{B}W_{t+1}P_t^{-1}W_t, \tag{29}$$

where the two auxiliary variables W_t and P_t are given by

$$W_t = \left(\widetilde{A} - \widetilde{B}L_t^{\mathbf{a}}\right)W_{t+1},\ W_T = 1, \tag{30}$$

$$P_t = P_{t+1} - W_{t+1}^2\widetilde{B}^2\left(f_{\bar{u}\bar{u}} + \widetilde{B}^2S_{t+1}\right)^{-1},\ P_T = 0. \tag{31}$$

We have now developed a linkage between the first-order approximation of the control function and a linear quadratic problem via a set of transformations. Having found a recursive solution of the linear quadratic problem we can back out the first-order approximation of the general problem by applying the set of transformations in reverse.

The optimal solution to the problem (LQP) is given by $\widetilde{u}_t = -(L_t^{\mathbf{a}} - L_t^{\mathbf{b}})\widetilde{x}_t$. Using the definitions $\widetilde{u}_t = \beta^{t/2}d\widetilde{u}_t$ and $\widetilde{x}_t = \beta^{t/2}dx_t$ yields

$$d\widetilde{u}_t = -(L_t^{\mathbf{a}} - L_t^{\mathbf{b}})dx_t.$$

Further, substituting $d\widetilde{u}_t = (du_t + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}dx_t)$ yields the optimal control of the problem (DLQP)

$$du_t = -(L_t^{\mathbf{a}} - L_t^{\mathbf{b}} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})dx_t.$$

Since increments are made around the steady state, $du_t = u_t - \bar{u}$ and $dx_t = x_t - \bar{x}$, the first-order approximated control function of the optimal control problem can be expressed by

$$u_t = \bar{u} - (L_t^{\mathbf{a}} - L_t^{\mathbf{b}} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})(x_t - \bar{x}), \tag{32}$$

where $L_t^{\mathbf{a}}$ and $L_t^{\mathbf{b}}$ are given by (27) - (31). This linearised control function ensures that the state reaches the steady state in the final period, i.e., restriction (3) holds also for this control function.[8]

[8] See Appendix 6.4.

6.3 Optimal Solution of the Linear Quadratic Problem

The solution to the linear quadratic problem (LQP) with the final state restricted to equal zero is derived in this section and the presentation closely parallels that of Section 4.5 in Lewis and Syrmos [13].

$$\text{(LQP)} \qquad \min_{\{\widetilde{u}_t\}_{t=0}^{\infty}} \frac{1}{2} \sum_{t=0}^{T-1} \{\widetilde{x}_t \widetilde{R} \widetilde{x}_t + \widetilde{u}_t f_{\bar{u}\bar{u}} \widetilde{u}_t\},$$

$$s.t. \quad \widetilde{x}_{t+1} = \widetilde{A}\widetilde{x}_t + \widetilde{B}\widetilde{u}_t \tag{33}$$

$$\widetilde{x}_T = 0. \tag{34}$$

The Lagrangian corresponding to the problem (LQP) is given by

$$\mathcal{L} = \widetilde{\mu}_{T+1}\widetilde{x}_T + \sum_{t=0}^{T-1} \frac{1}{2}(\widetilde{x}_t \widetilde{R} \widetilde{x}_t + \widetilde{u}_t f_{\bar{u}\bar{u}} \widetilde{u}_t) + \widetilde{\lambda}_{t+1}(\widetilde{A}\widetilde{x}_t + \widetilde{B}\widetilde{u}_t - \widetilde{x}_{t+1}).$$

First-order conditions imply

$$\widetilde{u}_t = -f_{\bar{u}\bar{u}}^{-1} \widetilde{B} \widetilde{\lambda}_{t+1} \tag{35}$$

$$\widetilde{\lambda}_t = R\widetilde{x}_t + \widetilde{B}\widetilde{\lambda}_{t+1}, \tag{36}$$

with boundary condition $\widetilde{\lambda}_T = \widetilde{\mu}_{T+1}$. We proceed by the sweep method which assumes that a linear relationship holds between the state and both Lagrangian multipliers at all time periods, i.e.,

$$\widetilde{\lambda}_t = S_t \widetilde{x}_t + W_t \widetilde{\mu}_{T+1}, \tag{37}$$

where S_t and W_t are left to be determined. Inserting (35) and (37) into (33) yields

$$\widetilde{x}_{t+1} = (1 + \widetilde{B}^2 f_{\bar{u}\bar{u}}^{-1} S_{t+1})^{-1}(\widetilde{A}\widetilde{x}_t - \widetilde{B}^2 f_{\bar{u}\bar{u}}^{-1} W_{t+1} \widetilde{\mu}_{T+1}). \tag{38}$$

Further, inserting this equation with (37) in (36) yields

$$\begin{aligned} &\left[-S_t + \widetilde{A}^2 S_{t+1}(1 + \widetilde{B}^2 f_{\bar{u}\bar{u}}^{-1} S_{t+1})^{-1} + \widetilde{R}\right] \widetilde{x}_t \\ &\qquad + \left[-W_t - \widetilde{A} S_{t+1}(1 + \widetilde{B}^2 f_{\bar{u}\bar{u}}^{-1} S_{t+1})^{-1} \widetilde{B}^2 f_{\bar{u}\bar{u}}^{-1} W_{t+1} + \widetilde{A} W_{t+1}\right] \widetilde{\mu}_{T+1} = 0. \end{aligned}$$

Since this equation must hold for all possible states the terms within both brackets must both be zero. Imposing this zero restriction yields first the Riccati equation (which corresponds to (6) when applying the matrix inversion lemma[9])

$$S_t = \widetilde{A}^2 S_{t+1}(1 + \widetilde{B}^2 f_{\bar{u}\bar{u}}^{-1} S_{t+1})^{-1} + \widetilde{R}, \tag{39}$$

Using the definition $L_t^{\mathbf{a}} = (f_{\bar{u}\bar{u}} + \widetilde{B}S_{t+1}\widetilde{B})^{-1}(\widetilde{B}S_{t+1}\widetilde{A})$ we can write the terms within the second bracket as

$$W_t = (\widetilde{A} - \widetilde{B}L_t^{\mathbf{a}})W_{t+1}, \tag{40}$$

with initial condition $W_T = 1$ which follows from the boundary condition above and (37). In order to determine the Lagrangian multiplier $\widetilde{\mu}_T$ we make the assumption that the final restriction can be represented as a linear function of the state and this multiplier,

$$\widetilde{x}_T = U_t \widetilde{x}_t + P_t \widetilde{\mu}_{T+1}, \tag{41}$$

where the auxiliary variables U_t and P_t are left to be determined. Trivially, at $t = T$ this condition holds if $P_T = 0$ and $U_T = 1$ which provides us with initial conditions. Taking the first difference yields

$$0 = U_{t+1}\widetilde{x}_{t+1} + P_{t+1}\widetilde{\mu}_{T+1} - U_t\widetilde{x}_t - P_t\ \widetilde{\mu}_{T+1}.$$

Inserting (38), applying the matrix inversion lemma, yields an expression in which the brackets necessarily must equal zero.

$$\begin{aligned}\Big[U_{t+1}\Big(\widetilde{A} - \widetilde{B}^2\widetilde{A}S_{t+1}(\widetilde{B}^2 S_{t+1} + f_{\bar{u}\bar{u}})^{-1}\Big) - U_t\Big]\,\widetilde{x}_t \\ + \Big[P_{t+1} - P_t - U_{t+1}\widetilde{B}^2 W_{t+1}(\widetilde{B}^2 S_{t+1} + f_{\bar{u}\bar{u}})^{-1}\Big]\,\widetilde{\mu}_{T+1} = 0.\end{aligned}$$

Setting the term within the first bracket to zero and applying the definition of $(L_t^{\mathbf{a}})$ it follows that $U_t = W_t$. Inserting this relationship into the second bracket yields

$$P_t = P_{t+1} - (W_{t+1}^2 \widetilde{B}^2(\widetilde{B}^2 S_{t+1} + f_{\bar{u}\bar{u}})^{-1}).$$

[9] The matrix inversion lemma, $(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$, implies $(1 + \widetilde{B}^2 f_{\bar{u}\bar{u}}^{-1} S_{t+1})^{-1} = 1 - \widetilde{B}^2 S_{t+1}(\widetilde{B}^2 S_{t+1} + f_{\bar{u}\bar{u}})^{-1}$.

The solution for $\widetilde{\mu}_T$ can now be seen from (41) which implies $\widetilde{\mu}_{T+1} = -P_t^{-1}W_t x_t$. We can now express the optimal control (35) as a function of the current state variable when inserting (33) and (37) which yields

$$\begin{aligned}\widetilde{u}_t &= -(\widetilde{B}^2 S_{t+1} + f_{\bar{u}\bar{u}})^{-1}\widetilde{B}(S_{t+1}\widetilde{A} - W_{t+1}P_t^{-1}W_t)\widetilde{x}_t \\ &= -(L_t^{\mathbf{a}} - L_t^{\mathbf{b}})\widetilde{x}_t,\end{aligned}$$

where $L_t^{\mathbf{b}} = (\widetilde{B}^2 S_{t+1} + f_{\bar{u}\bar{u}})^{-1}\widetilde{B}W_{t+1}P_t^{-1}W_t$.

6.4 First-order Approximation: $x_T = \bar{x}$

It can be instructive to note that the linearised control function (4) indeed ensures that the state reaches the steady state at time $t = T$. In the next to last period the control function simplifies. From (5) - (9) we have $L_{T-1}^{\mathbf{a}} = 0$ and $L_{T-1}^{\mathbf{b}} = -(AB^{-1} - f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})$, which leads to the simple structure

$$\begin{aligned}u_{T-1} &= \bar{u} - (-L_{T-1}^{\mathbf{b}} + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})(x_t - \bar{x}) \\ &= B^{-1}\bar{x} - AB^{-1}x_{T-1},\end{aligned} \tag{42}$$

where the second equality follows from the steady state control relation $\bar{u} = (1-A)B^{-1}\bar{x}$. The approximated control function as given by equation (42) thus ensures that the final condition (3) holds. From the transition equation (2) it follows that

$$x_T = Ax_{T-1} + Bu_{T-1} = Ax_{T-1} + B(B^{-1}\bar{x} - AB^{-1}x_{T-1}) = \bar{x}.$$

6.5 Proof: Theorem 3.1

The first-order approximation (4) consists of two sequences $L_t^{\mathbf{a}}$ and $L_t^{\mathbf{b}}$. We show the linkage between the generalised Fibonacci sequence and these sequences separately.

6.5.1 Fibonacci and Optimal Control: $L_t^{\mathbf{a}}$

First, we note that the ratio of Fibonacci numbers, $\mathcal{H}_n = \mathcal{F}_{n-1}/\mathcal{F}_n$, can also be generated by

$$\mathcal{H}_{n+2} = \frac{a + b_{n+1}\mathcal{H}_n}{a^2 + b_{n+2} + ab_{n+1}\mathcal{H}_n}, \tag{43}$$

with initial value $\mathcal{H}_1 = 0$. Further, combining (5) with (6) we can write $S_{t+1} = f_{\bar{u}\bar{u}}\widetilde{A}\widetilde{B}^{-1}L^{\mathbf{a}}_{t+1} + \widetilde{R}$, which when inserted into (5) yields

$$\widetilde{A}^{-1}L^{\mathbf{a}}_t = \frac{\widetilde{B} + f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\widetilde{A}^2(\widetilde{A}^{-1}L^{\mathbf{a}}_{t+1})}{\widetilde{B}^2 + f_{\bar{u}\bar{u}}\widetilde{R}^{-1} + \widetilde{B}f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\widetilde{A}^2(\widetilde{A}^{-1}L^{\mathbf{a}}_{t+1})}. \tag{44}$$

Comparing (43) with (44) we note that using the particular values $a = \widetilde{B}$ and $b_{n+2} = f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\widetilde{A}^2$ when n is even and $b_{n+2} = f_{\bar{u}\bar{u}}\widetilde{R}^{-1}$ when n is odd makes (43) identical with the sequence of the transformed feedback (44) with appropriate change of index. The sequence $\widetilde{A}^{-1}L^{\mathbf{a}}_t$ runs backward from an initial value at time $t = T - 1$. If we make the index change $n = 2(T - t) - 1$, the sequence $\mathcal{H}_n = \mathcal{H}_{2(T-t)-1}$ begins at the initial value $\mathcal{H}_1 = 0$. Since from (5) the initial value of the feedback equation is zero, and consequently $\widetilde{A}^{-1}L_{T-1} = 0$, we have derived the following relationship

$$L^{\mathbf{a}}_t = \widetilde{A}\,\mathcal{H}_{2(T-t)-1}, \qquad 0 \le t \le T - 1. \tag{45}$$

6.5.2 Fibonacci and Optimal Control: $L^{\mathbf{b}}_t$

In order to derive the relationship between the second part of the control function ($L^{\mathbf{b}}_t$) and the generalised Fibonacci sequence we note that the inverse of (43) can be written

$$\mathcal{H}^{-1}_{n+2} = \frac{(a^2 + b_{n+2})\mathcal{H}^{-1}_n + ab_{n+1}}{a\mathcal{H}^{-1}_n + b_{n+1}}. \tag{46}$$

Multiplying the Riccati equation (6) by $(\widetilde{B}\widetilde{R}^{-1})$ yields

$$(\widetilde{B}\widetilde{R}^{-1}S_t) = \frac{(\widetilde{B}^2 + f_{\bar{u}\bar{u}}R^{-1}\widetilde{A}^2)(\widetilde{B}\widetilde{R}^{-1}S_{t+1}) + B\widetilde{R}^{-1}f_{\bar{u}\bar{u}}}{\widetilde{B}(\widetilde{B}\widetilde{R}^{-1}S_{t+1}) + \widetilde{R}^{-1}f_{\bar{u}\bar{u}}}. \tag{47}$$

We note that the same choice of coefficients as in section 6.5.1 makes the sequence (46) identical to the sequence (47), i.e., $a = \widetilde{B}$ and $b_{n+2} = f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\widetilde{A}^2$ when n is even and $b_{n+2} = f_{\bar{u}\bar{u}}\widetilde{R}^{-1}$ when n is odd. The sequence $(\widetilde{B}\widetilde{R}^{-1}S_t)$ runs backward from time $(t = T)$ with an initial condition which follows from the Riccati equation $(\widetilde{B}\widetilde{R}^{-1}S_T) = 0$. Since $\mathcal{F}_0/\mathcal{F}_{-1} = 0$, we define $\mathcal{H}^{-1}_0 := 0$, even though $\mathcal{H}_0$ is undefined. This gives the following relationship between the solution of the Riccati equation and the ratio of Fibonacci sequences, for $0 \le t \le T$,

$$S_t = \widetilde{R}\widetilde{B}^{-1}\mathcal{H}^{-1}_{2(T-t)}. \tag{48}$$

Further, we note that from (8) and (9)

$$
\begin{aligned}
W_{T-1} &= \widetilde{A}\left(1-\widetilde{B}\mathcal{H}_1\right)W_T = \widetilde{A},\\
P_{T-1} &= P_T - \frac{W_T^2\widetilde{B}^2}{f_{\bar{u}\bar{u}}+\widetilde{B}^2 S_T} = -\frac{\widetilde{B}^2}{f_{\bar{u}\bar{u}}},
\end{aligned}
$$

hence the initial condition $L^{\mathbf{b}}_{T-1}$ is, from (7)

$$
L^{\mathbf{b}}_{T-1} = \left(f_{\bar{u}\bar{u}}+\widetilde{B}^2 S_T\right)^{-1}\widetilde{B}W_T P_{T-1}^{-1}W_{T-1} = -\frac{\widetilde{A}\widetilde{B}}{f_{\bar{u}\bar{u}}\frac{\widetilde{B}^2}{f_{\bar{u}\bar{u}}}} = -\frac{\widetilde{A}}{\widetilde{B}}.
$$

For $k = 0, 1, 2, 3, \dots,$ we can rewrite the sequence of generalised Fibonacci numbers $(\mathcal{F}_n)$

$$
\begin{aligned}
\mathcal{F}_{2k+2} &= \widetilde{B}\mathcal{F}_{2k+1} + \frac{\widetilde{A}^2 f_{\bar{u}\bar{u}}}{\widetilde{R}}\mathcal{F}_{2k},\ \mathcal{F}_0 = 0,\ \mathcal{F}_1 = 1, &(49)\\
\mathcal{F}_{2k+1} &= \widetilde{B}\mathcal{F}_{2k} + \frac{f_{\bar{u}\bar{u}}}{\widetilde{R}}\mathcal{F}_{2k-1},\ \mathcal{F}_0 = 0,\ \mathcal{F}_{-1} = \frac{\widetilde{R}}{f_{\bar{u}\bar{u}}}. &(50)
\end{aligned}
$$

With these premises, we want to show that also the second feedback coefficient can be explicitly expressed in terms of generalised Fibonacci numbers; more specific, we have that

$$
\begin{aligned}
W_{T-k} &= \left(\frac{\widetilde{A}f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{k-1}\frac{\widetilde{A}}{\mathcal{F}_{2k-1}},\ k = 0, 1, 2, \dots, T, &(51)\\
P_{T-k} &= -\frac{\widetilde{B}}{f_{\bar{u}\bar{u}}}\mathcal{H}_{2k}^{-1},\ k = 0, 1, 2, \dots, T, &(52)\\
L^b_{T-k} &= -\left(\frac{\widetilde{A}f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{2(k-1)}\frac{\widetilde{A}}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}},\ k = 1, 2, \dots, T, &(53)
\end{aligned}
$$

To this end, we use the principle of induction. Having in mind that

$$
\begin{aligned}
\mathcal{F}_{-1} &= \frac{\widetilde{R}}{f_{\overline{u}\overline{u}}},\\
\mathcal{H}_0^{-1} &= 0,
\end{aligned}
$$

we see that the initial conditions are satisfied since

$$\begin{aligned}
W_T &= \left(\frac{\widetilde{A}f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{-1}\frac{\widetilde{A}}{\frac{\widetilde{R}}{f_{\bar{u}\bar{u}}}} = 1,\\
P_T &= -\frac{\widetilde{B}}{f_{\bar{u}\bar{u}}}\mathcal{H}_0^{-1} = 0,\\
L^{\mathbf{b}}_{T-1} &= -\left(\frac{\widetilde{A}f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{0}\frac{\widetilde{A}}{\mathcal{F}_1\mathcal{F}_2} = -\frac{\widetilde{A}}{\widetilde{B}}.
\end{aligned} \tag{54}$$

Assume now that expressions (51 - 53) are true for $k = p$. We want to show that this implies that they also are true for $k = p + 1$. Indeed, equation (8) with (45) yields that

$$\begin{aligned}
W_{T-(p+1)} &= \widetilde{A}\left(1-\widetilde{B}\mathcal{H}_{2p+1}\right)W_{T-p} = \widetilde{A}^2\left(\frac{\widetilde{A}f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{p-1}\left(1-\widetilde{B}\frac{\mathcal{F}_{2p}}{\mathcal{F}_{2p+1}}\right)\frac{1}{\mathcal{F}_{2p-1}}\\
&= \left(\frac{\widetilde{A}f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{p}\frac{\widetilde{R}}{f_{\bar{u}\bar{u}}}\frac{\mathcal{F}_{2p+1}-\widetilde{B}\mathcal{F}_{2p}}{\mathcal{F}_{2p-1}}\frac{\widetilde{A}}{\mathcal{F}_{2p+1}} = \left(\frac{\widetilde{A}f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{p}\frac{\widetilde{A}}{\mathcal{F}_{2(p+1)-1}},
\end{aligned}$$

where the last equality follows from (50). Moreover, equation (9) with (48) yields that

$$\begin{aligned}
P_{T-(p+1)} &= P_{T-p} - \widetilde{B}^2W_{T-p}^2\left(f_{\overline{uu}} + \widetilde{R}\widetilde{B}\mathcal{H}_{2p}^{-1}\right)^{-1}\\
&= -\frac{\widetilde{B}}{f_{\bar{u}\bar{u}}}\mathcal{H}_{2p}^{-1} - \left(\frac{\widetilde{A}f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{2(p-1)}\frac{\widetilde{B}^2\widetilde{A}^2}{\mathcal{F}_{2p-1}^2\left(f_{\bar{u}\bar{u}}+\widetilde{R}\widetilde{B}\frac{\mathcal{F}_{2p}}{\mathcal{F}_{2p-1}}\right)}\\
&= -\frac{\widetilde{B}}{f_{\bar{u}\bar{u}}}\left(\frac{\mathcal{F}_{2p}}{\mathcal{F}_{2p-1}} + \left(\frac{\widetilde{A}f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{2p-1}\frac{\widetilde{A}\widetilde{B}}{\mathcal{F}_{2p-1}\mathcal{F}_{2p+1}}\right)\\
&= -\frac{\widetilde{B}}{f_{\bar{u}\bar{u}}}\left(\frac{\mathcal{F}_{2p}}{\mathcal{F}_{2p-1}} + \frac{\mathcal{F}_{2p+2}\mathcal{F}_{2p-1}-\mathcal{F}_{2p}\mathcal{F}_{2p+1}}{\mathcal{F}_{2p-1}\mathcal{F}_{2p+1}}\right)\\
&= -\frac{\widetilde{B}}{f_{\bar{u}\bar{u}}}\frac{\mathcal{F}_{2(p+1)}}{\mathcal{F}_{2p+1}} = -\frac{\widetilde{B}}{f_{\bar{u}\bar{u}}}\mathcal{H}_{2(p+1)}^{-1},
\end{aligned}$$

where we have used the relation (55), corresponding to d'Ocagne's identity for regular Fibonacci numbers. Hence expressions (51) and (52) follow by the induction principle. Finally, expression (7) together with (48) for $k = 2, 3, \ldots, T$, gives

$$
\begin{aligned}
L^{\mathbf{b}}_{T-k} &= \left(f_{\bar{u}\bar{u}} + \widetilde{R}\widetilde{B}\mathcal{H}^{-1}_{2(k-1)}\right)^{-1} \widetilde{B} W_{T-(k-1)} P^{-1}_{T-k} W_{T-k} \\
&= -\frac{\widetilde{B}\left(\frac{\widetilde{A} f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{k-2} \widetilde{A}\left(\frac{\widetilde{A} f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{k-1} \widetilde{A}}{\frac{\widetilde{B}}{f_{\bar{u}\bar{u}}}\mathcal{H}^{-1}_{2k}\left(f_{\overline{uu}} + \widetilde{R}\widetilde{B}\frac{\mathcal{F}_{2(k-1)}}{\mathcal{F}_{2k-3}}\right)\mathcal{F}_{2k-3}\mathcal{F}_{2k-1}} \\
&= -\frac{f_{\bar{u}\bar{u}}\widetilde{A}^2\left(\frac{\widetilde{A} f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{2k-3}}{\mathcal{F}_{2k}\left(f_{\bar{u}\bar{u}}\mathcal{F}_{2k-3} + \widetilde{R}\widetilde{B}\mathcal{F}_{2(k-1)}\right)} = \left(\frac{\widetilde{A} f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{2(k-1)} \frac{\widetilde{A}}{\mathcal{F}_{2k}\mathcal{F}_{2k-1}},
\end{aligned}
$$

where the last equality follows from (50).

In proving the explicit expression for P_{T-k}, we used the identity

$$
\mathcal{F}_{2k+2}\mathcal{F}_{2k-1} - \mathcal{F}_{2k}\mathcal{F}_{2k+1} = \widetilde{A}\widetilde{B}\left(\frac{\widetilde{A} f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{2k-1}, \; k = 0, 1, 2, \ldots. \tag{55}
$$

This identity is also proved by using induction. First, we note that the initial condition is satisfied since

$$
\mathcal{F}_2\mathcal{F}_{-1} - \mathcal{F}_0\mathcal{F}_1 = \widetilde{B}\frac{\widetilde{R}}{f_{\overline{uu}}} - 0 \cdot 1 = \widetilde{A}\widetilde{B}\left(\frac{\widetilde{A} f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{-1}.
$$

Now, let us assume that the identity is true for $k = p$, that is,

$$
\mathcal{F}_{2p+2}\mathcal{F}_{2p-1} - \mathcal{F}_{2p}\mathcal{F}_{2p+1} = \widetilde{A}\widetilde{B}\left(\frac{\widetilde{A} f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{2p-1}.
$$

The proof is complete if we can show that it also holds for $k = p + 1$. Indeed,

$$
\begin{aligned}
&\mathcal{F}_{2p+4}\mathcal{F}_{2p+1} - \mathcal{F}_{2p+2}\mathcal{F}_{2p+3} \\
&= \left(\widetilde{B}\mathcal{F}_{2p+3} + \frac{\widetilde{A}^2 f_{\bar{u}\bar{u}}}{\widetilde{R}}\mathcal{F}_{2p+2}\right)\mathcal{F}_{2p+1} - \left(\widetilde{B}\mathcal{F}_{2p+1} + \frac{\widetilde{A}^2 f_{\bar{u}\bar{u}}}{\widetilde{R}}\mathcal{F}_{2p}\right)\mathcal{F}_{2p+3} \\
&= \frac{\widetilde{A}^2 f_{\bar{u}\bar{u}}}{\widetilde{R}}\left(\mathcal{F}_{2p+2}\mathcal{F}_{2p+1} - \mathcal{F}_{2p}\mathcal{F}_{2p+3}\right) \\
&= \frac{\widetilde{A}^2 f_{\bar{u}\bar{u}}}{\widetilde{R}}\left(\mathcal{F}_{2p+2}\left(\widetilde{B}\mathcal{F}_{2p} + \frac{f_{\bar{u}\bar{u}}}{\widetilde{R}}\mathcal{F}_{2p-1}\right) - \mathcal{F}_{2p}\left(\widetilde{B}\mathcal{F}_{2p+2} + \frac{f_{\bar{u}\bar{u}}}{\widetilde{R}}\mathcal{F}_{2p+1}\right)\right) \\
&= \frac{\widetilde{A}^2 f^2_{\bar{u}\bar{u}}}{\widetilde{R}^2}\left(\mathcal{F}_{2p+2}\mathcal{F}_{2p-1} - \mathcal{F}_{2p}\mathcal{F}_{2p+1}\right) = \widetilde{A}\widetilde{B}\left(\frac{\widetilde{A} f_{\bar{u}\bar{u}}}{\widetilde{R}}\right)^{2p+1},
\end{aligned}
$$

where the last equality follows from the induction assumption. Changing index, we have thus shown how the Fibonacci sequence enters the second feedback term

$$L_t^{\mathbf{b}} = -\frac{\widetilde{A}\left(\widetilde{A}f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\right)^{2(T-t-1)}}{\mathcal{F}_{2(T-t)-1}\mathcal{F}_{2(T-t)}}, \quad 0 \le t \le T-1. \tag{56}$$

6.6 Proof: Corollary 3.1

Since

$$\begin{aligned} L_{T-k}^{\mathbf{a}} &= \widetilde{A}\mathcal{H}_{2k-1}, \\ L_{T-k}^{\mathbf{b}} &= -\left(\frac{\widetilde{A}f_{\overline{uu}}}{\widetilde{R}}\right)^{2(k-1)} \frac{\widetilde{A}}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}}, \end{aligned}$$

we see that we can simplify

$$\widetilde{u}_{T-k} = -\left(L_{T-k}^{\mathbf{a}} - L_{T-k}^{\mathbf{b}}\right)\widetilde{x}_{T-k},$$

when $\widetilde{A}^2 = 1$. First, note that

$$\begin{aligned} L_{T-k}^{\mathbf{a}} - L_{T-k}^{\mathbf{b}} &= \widetilde{A}\left(\mathcal{H}_{2k-1} + \frac{\left(\frac{\widetilde{A}f_{\overline{uu}}}{\widetilde{R}}\right)^{2(k-1)}}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}}\right) \\ &= \widetilde{A}\frac{\mathcal{F}_{2k-2}\mathcal{F}_{2k} + \left(\widetilde{A}^2\right)^{k-1}\left(\frac{\widetilde{A}f_{\overline{uu}}}{\widetilde{R}}\right)^{2(k-1)}}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}}. \end{aligned}$$

If we let $\widetilde{A}^2 = 1$, we then get that

$$L_{T-k}^{a} - L_{T-k}^{b} = \widetilde{A}\frac{\mathcal{F}_{2k-1}^2}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}} = \widetilde{A}\mathcal{H}_{2k},$$

by noting that

$$\mathcal{F}_{2k-1}^2 - \mathcal{F}_{2k-2}\mathcal{F}_{2k} = \left(\frac{f_{\overline{uu}}}{\widetilde{R}}\right)^{2(k-1)}, \; k = 1, 2, 3, \ldots,$$

which follows from setting $n = 2k-1$ in the identity

$$\mathcal{F}_n^2 - \mathcal{F}_{n-1}\mathcal{F}_{n+1} = \left(-\frac{f_{\overline{uu}}}{\widetilde{R}}\right)^{n-1}, \; n = 1, 2, 3, \ldots. \tag{57}$$

This identity is proved by using induction. First, we note that the initial condition is satisfied since

$$\mathcal{F}_1^2 - \mathcal{F}_0\mathcal{F}_2 = 1 = \left(-\frac{f_{\overline{uu}}}{\widetilde{R}}\right)^0.$$

Now, let us assume that the identity is true for $k = p$, that is,

$$\mathcal{F}_p^2 - \mathcal{F}_{p-1}\mathcal{F}_{p+1} = \left(-\frac{f_{\overline{uu}}}{\widetilde{R}}\right)^{p-1}.$$

The proof is complete if we can show that it then also holds for $k = p+1$. Indeed, by using $\widetilde{A}^2 = 1$, we get

$$\mathcal{F}_{p+1}^2 - \mathcal{F}_p\mathcal{F}_{p+2} = \left(\widetilde{B}\mathcal{F}_p + \frac{f_{\overline{uu}}}{\widetilde{R}}\mathcal{F}_{p-1}\right)\mathcal{F}_{p+1} - \left(\widetilde{B}\mathcal{F}_{p+1} + \frac{f_{\overline{uu}}}{\widetilde{R}}\mathcal{F}_p\right)\mathcal{F}_p$$
$$= \frac{f_{\overline{uu}}}{\widetilde{R}}\left(\mathcal{F}_{p+1}\mathcal{F}_{p-1} - \mathcal{F}_p^2\right) = \left(-\frac{f_{\overline{uu}}}{\widetilde{R}}\right)\left(-\frac{f_{\overline{uu}}}{\widetilde{R}}\right)^{p-1} = \left(-\frac{f_{\overline{uu}}}{\widetilde{R}}\right)^{p}$$

where the penultimate equality follows from the induction assumption.

Remark 6.1 *Identity (57) is a generalisation of Cassini's identity*

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}, \; n = 1, 2, 3, \ldots,$$

for regular Fibonacci numbers.

Hence in this special case, we have that

$$\widetilde{u}_{T-k} = -\left(L_{T-k}^a - L_{T-k}^b\right)\widetilde{x}_{T-k} = -\widetilde{A}\mathcal{H}_{2k}\widetilde{x}_{T-k},$$

or

$$\widetilde{u}_t = -\left(L_t^{\mathbf{a}} - L_t^{\mathbf{b}}\right)\widetilde{x}_t = -\widetilde{A}\mathcal{H}_{2(T-t)}\widetilde{x}_t.$$

6.7 Example: Narrative Details on the Brock-Mirman Model

The Brock-Mirman model considers a representative household maximising utility subject to economic constraints. In particular, it considers an economy where the total amount of goods (y_t) is produced using capital (x_t) as input in the production process, i.e.,

$$y_t = \gamma x_t^{\alpha}, \tag{58}$$

where $\gamma > 0$ and $0 < \alpha < 1$. In a closed economy, what is produced in a given year must either be consumed (c_t) or invested (u_t) as given by the national accounts identity

$$y_t = c_t + u_t. \tag{59}$$

Further, if we make the simplifying assumption that capital fully depreciates, the consecutive level of capital will equal current investments, i.e.,

$$x_{t+1} = u_t. \tag{60}$$

Given an initial level of capital (x_0), the objective of the representative household is to maximise a discounted ($0 < \beta < 1$) sum of utilities

$$\sum_{t=0}^{T-1} \beta^t \ln(c_t), \tag{61}$$

subject to the three economic constraints (58) - (60) and subject to capital reaching the steady state value in the final time period

$$x_T = \bar{x}. \tag{62}$$

The form of the Brock-Mirman model as given in the main text follows by inserting both the production function (58) and the national accounts identity (59) into the objective function (61). More details on the Brock-Mirman model can be found in Section 3.1.2 in Ljungqvist and Sargent [12].

6.8 Example: Deriving the Steady State

In this section we derive the steady state of the Brock-Mirman model. Define the Hamiltonian

$$\mathrm{H} := -\beta^t \ln(\gamma x_t^\alpha - u_t) + \lambda_{t+1} u_t,$$

where λ_{t+1} is the multiplier. The first-order conditions are[10]

$$\begin{aligned} \mathrm{H}_{u_t} = 0 \qquad &\Rightarrow \qquad -\beta^t (\gamma x_t^\alpha - u_t)^{-1} = \lambda_{t+1} \\ \lambda_t = \mathrm{H}_{x_t} \qquad &\Rightarrow \qquad \lambda_t = -\beta^t \alpha \gamma x_t^{\alpha-1} (\gamma x_t^\alpha - u_t)^{-1}, \end{aligned}$$

Combining these first-order conditions, letting $c_t = \gamma x_t^\alpha - u_t$, yields the Euler-Lagrange equation

$$c_{t+1} = c_t \beta \alpha \gamma x_{t+1}^{\alpha-1}.$$

[10]See Section 12.4 in Sydsaeter et al. [16].

At the steady state both the control and the state remains unchanged, $\bar{c} = c_t = c_{t+1}$ and $\bar{x} = x_t = x_{t+1}$. The Euler equation can thus be solved to yield

$$\bar{x} = (\alpha\beta\gamma)^{1/(1-\alpha)}.$$

Further, the steady state levels of investment and consumption are given by

$$\bar{u} = \bar{x}, \qquad \bar{c} = \gamma\bar{x}^{\alpha} - \bar{x}.$$

6.9 Example: Second Derivatives

In this section we provide the second derivatives of the criterion function evaluated at the steady state. In particular, we have

$$\begin{aligned} f_{\bar{x}\bar{x}} &= \alpha^2\gamma^2\bar{c}^{(-2)}\bar{x}^{2(\alpha-1)} + \alpha(1-\alpha)\gamma\bar{c}^{(-1)}\bar{x}^{(\alpha-2)} \\ f_{\bar{u}\bar{u}} &= \bar{c}^{(-2)} \\ f_{\bar{x}\bar{u}} &= -\alpha\gamma\bar{c}^{(-2)}\bar{x}^{\alpha-1}. \end{aligned}$$

From Appendix 6.8 we let $\bar{c} = (1-\alpha)\alpha^{-1}$ when imposing the restrictions $\beta = 1$ and $\gamma = \alpha^{-1}$. Also inserting the normalisation $\bar{x} = 1$ and $\alpha = 1 - \phi^{-1}$ yields the results given in the main text

$$f_{\bar{x}\bar{x}} = 2(1-\phi^{-1}), \quad f_{\bar{u}\bar{u}} = 1-\phi^{-1}, \quad f_{\bar{x}\bar{u}} = -(1-\phi^{-1}).$$

In order to derive $f_{\bar{x}\bar{x}}$ we have used the property

$$\alpha(1-\alpha)^{-2} = (1-\phi^{-1})\phi^2 = (\phi-1)\phi = 1.$$

where the last equality follows from applying (11).

6.10 Example: Generalised Fibonacci Sequence

In this section we illustrate that the Fibonacci sequence entering the control function of the Brock-Mirman model is the original Fibonacci sequence. The generalised Fibonacci sequence is in this example defined by

$$\mathcal{F}_{n+2} = a\mathcal{F}_{n+1} + b_{n+2}\mathcal{F}_n, \quad \mathcal{F}_0 = 0, \quad \mathcal{F}_1 = 1,$$

with the particular coefficients $a = \beta^{1/2}B$ and

$$b_{n+2} = f_{\bar{u}\bar{u}}(f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}}f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})^{-1}\beta(A - Bf_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})^2,$$

when n is even and $b_{n+2} = f_{\bar{u}\bar{u}}(f_{\bar{x}\bar{x}} - f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}^2)^{-1}$ when n is odd. It follows immediately that $a = 1$. To find the expression for b_{n+2} we need the second derivatives of the criterion function. From Appendix 6.9 it follows that $f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}} = -1$ and $f_{\bar{x}\bar{x}} - f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}^2 = (1 - \phi^{-1})$. Using the parameter values $A = 0$ and $\beta = B = 1$ yields $b_{n+2} = 1$.

6.11 Section 4: Explicit Solutions of Odd and Even Indexed Fibonacci Sequences

Since (15) has constant coefficients there is an explicit solution describing this sequence. Consider the recurrence equation

$$\begin{aligned} g_{n+2} &:= c_1 g_{n+1} + c_2 g_n && (63)\\ &= c_1(c_1 g_n + c_2 g_{n-1}) + c_2 g_n \\ &= (c_1^2 + c_2)g_n + c_1 c_2(c_1^{-1}(g_n - c_2 g_{n-2})) \\ &= (c_1^2 + 2c_2)g_n - c_2^2 g_{n-2}. && (64) \end{aligned}$$

We note that (64) describes the sequence (15) when coefficients are matched, i.e.,

$$c_1 = \sqrt{\widetilde{B}^2 + f_{\bar{u}\bar{u}}\widetilde{R}^{-1}(1 - \widetilde{A})^2}, \qquad c_2 = f_{\bar{u}\bar{u}}\widetilde{R}^{-1}\widetilde{A}. \tag{65}$$

The general solution to the sequence g_n is well known and depends on whether the characteristic equation $r^2 = c_1 r + c_2$ has two real roots, one real root or no real roots. Due to the assumption of a convex criterion function f in the optimal control problem it is only the real solution which is relevant, i.e., $r_{1,2} = (c_1 \pm \sqrt{c_1^2 + 4c_2})/2$.[11] Given two initial values the general solution is then given by

$$g_n = Gr_1^n + Kr_2^n, \tag{66}$$

where G and K are constants to be determined from the initial conditions. These initial conditions depend on whether we are considering the odd or even indexed Fibonacci sequence.

Let g_n^e denote the solution to equation (66) with initial values corresponding to even indexed Fibonacci numbers. For example, consider the initial value of the sequence $\mathcal{F}_{2(T-t-1)}$ when time is running backwards

[11] See Appendix 6.12.

from $t = T-1$, i.e., $g^e_0 = \mathcal{F}_0$ and $g^e_2 = \mathcal{F}_2$. Given these initial conditions, solving for the constants G and K in (66), gives the solution[12]

$$g^e_n = \tilde{B}\,\frac{r_1^n - r_2^n}{r_1^2 - r_2^2}.$$

In terms of the Fibonacci sequence, $g^e_n = \mathcal{F}_n$ when n is even. The explicit expressions for the even indexed Fibonacci sequences entering the control function are then given by

$$\mathcal{F}_{2(T-t-1)} = g^e_{2(T-t-1)} = \widetilde{B}\,\frac{r_1^{2(T-t-1)} - r_2^{2(T-t-1)}}{r_1^2 - r_2^2} \tag{67}$$

$$\mathcal{F}_{2(T-t)} = g^e_{2(T-t)} = \widetilde{B}\,\frac{r_1^{2(T-t)} - r_2^{2(T-t)}}{r_1^2 - r_2^2}. \tag{68}$$

Correspondingly, we let g^o_n denote the solution to equation (66) with initial values corresponding to the initial value of the odd indexed Fibonacci numbers, i.e., $g^o_1 = \mathcal{F}_1$ and $g^o_{-1} = \mathcal{F}_{-1}$. Given these initial conditions, solving for the constants G and K in (66), gives the solution[13]

$$g^o_n = \frac{(r_1 + \tilde{A}r_2)r_1^n - (r_2 + \tilde{A}r_1)r_2^n}{r_1^2 - r_2^2}.$$

Since $g^o_n = \mathcal{F}_n$ when n is odd, the explicit solution of the odd indexed Fibonacci sequence is then given by

$$\mathcal{F}_{2(T-t)-1} = \frac{(r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1}}{r_1^2 - r_2^2}. \tag{69}$$

6.12 Section 4: Both Roots Are Real

This section shows that the both roots of the characteristic equation corresponding to the solution of every second generalised Fibonacci sequence are real due to the convexity assumption of the criterion function f. The general solution to the sequence

$$g_{n+2} = c_1 g_{n+1} + c_2 g_n,$$

is well known and depends on whether the characteristic equation $r^2 = c_1 r + c_2$ has two real roots, one real root or no real roots. Due to the assumption of a convex criterion function f in the optimal control problem we show that it is only the real solution which is relevant, i.e., $c_1^2 + 4c_2 > 0$. Indeed, given the expressions

[12] See Appendix 6.13.

[13] See Appendix 6.13.

for c_1 and c_2 it follows that

$$c_1^2 + 4c_2 = \widetilde{B}^2 + \frac{f_{\bar{u}\bar{u}}}{\widetilde{R}} \left(1 - \widetilde{A}\right)^2 + 4\frac{f_{\bar{u}\bar{u}}}{\widetilde{R}} \widetilde{A} = \widetilde{B}^2 + \frac{f_{\bar{u}\bar{u}}}{\widetilde{R}} \left(1 + \widetilde{A}\right)^2 > 0,$$

if $\frac{f_{\bar{u}\bar{u}}}{\widetilde{R}} > 0$. This holds since

$$\frac{f_{\bar{u}\bar{u}}}{\widetilde{R}} = \frac{f_{\bar{u}\bar{u}}}{f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}} f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}}} = \frac{f_{\bar{u}\bar{u}}^2}{f_{\bar{u}\bar{u}} f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}}^2} > 0,$$

which is positive from the convexity assumption, i.e., the Hessian is positive

$$\begin{vmatrix} f_{\bar{x}\bar{x}} & f_{\bar{x}\bar{u}} \\ f_{\bar{x}\bar{u}} & f_{\bar{u}\bar{u}} \end{vmatrix} = f_{\bar{u}\bar{u}} f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}}^2 > 0.$$

6.13 Section 4: The General Solution: g_n

The general solution to the difference equation

$$g_{n+2} = c_1 g_{n+1} + c_2 g_n, \tag{70}$$

when both roots of the characteristic equation $r^2 = c_1 r + c_2$ are real, is given by

$$g_n = G r_1^n + K r_2^n, \tag{71}$$

where the constants G and K are determined by initial conditions. We consider the case of even and odd indexed Fibonacci sequences separately, i.e., we find the sequences $g_n^e = \mathcal{F}_n$ when n is even and $g_n^o = \mathcal{F}_n$ when n is odd.[14]

6.13.1 Section 4: The Even Indexed Sequence: g_n^e

From the generalised Fibonacci sequence, $g_0^e = \mathcal{F}_0 = 0$, and

$$g_2^e = \mathcal{F}_2 = a_2 \mathcal{F}_1 + b_2 \mathcal{F}_0 = a_2 = \widetilde{B},$$

[14]The superscript e emphasise that the sequence relates to the even indexed Fibonacci sequence (g^e, G^e and K^e) while the superscript o refers to the odd indexed sequence.

which, when inserted into (70), gives the initial condition

$$g_1^e = c_1^{-1}(g_2^e - c_2 g_0^e) = c_1^{-1} g_2^e = \frac{\widetilde{B}}{r_1 + r_2},$$

where we have used the property $c_1 = r_1 + r_2$. From the general solution (71) we get

$$g_0^e = 0 = G^e r_1^0 + K^e r_2^0 \quad \Rightarrow \quad G^e = -K^e.$$

Inserting this result when applying the second initial condition g_1^e yields

$$g_1^e = G^e r_1 + K^e r_2 = K^e(r_2 - r_1) = \frac{\widetilde{B}}{r_1 + r_2}.$$

Together, this implies

$$G^e = \frac{\widetilde{B}}{r_1^2 - r_2^2}, \qquad K^e = -\frac{\widetilde{B}}{r_1^2 - r_2^2}.$$

Using these results in the general solution (71) gives

$$g_n^e = \tilde{B}\,\frac{r_1^n - r_2^n}{r_1^2 - r_2^2}.$$

6.13.2 Section 4: The Odd Indexed Sequence: g_n^o

From the generalised Fibonacci sequence we have $g_1^o = \mathcal{F}_1 = 1$ and $g_{-1}^o = \mathcal{F}_{-1} = \widetilde{R} f_{\bar{u}\bar{u}}^{-1}$ which gives

$$g_0^o = (1 - \widetilde{A}) c_1^{-1},$$

when inserted into (70) and applying the matched coefficient $c_2 = f_{\bar{u}\bar{u}} \widetilde{R}^{-1} \widetilde{A}$. In order to determine G^o and K^o we use the initial values g_0^o and g_1^o. From the general solution (71) we get

$$g_0^o = (1 - \widetilde{A}) c_1^{-1} = G^o r_1^0 + K^o r_2^0,$$

or

$$G^o = (1 - \widetilde{A}) c_1^{-1} - K^o. \tag{72}$$

From the other initial condition, we get

$$g_1^o = G^o r_1 + K^o r_2 = 1,$$

which, when inserting (72) and using the relation $c_1 = r_1 + r_2$, yields

$$K^o = -\frac{r_2 + \widetilde{A} r_1}{r_1^2 - r_2^2}.$$

Inserting this result back into (72) gives

$$G^o = \frac{r_1 + \widetilde{A} r_2}{r_1^2 - r_2^2}.$$

Using these results in the general solution (71) gives

$$g_n^o = G^o r_1^n + K^o r_2^n = \frac{(r_1 + \tilde{A} r_2) r_1^n - (r_2 + \tilde{A} r_1) r_2^n}{r_1^2 - r_2^2}.$$